Loci in the Argand Diagram

Starter

1. (Review of last lesson) Convert $1 - \text{i}$ to $[r,\ \theta]$ form.

Working: $r = \left|1 - \text{i}\right| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

Ignore the signs: $\tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

$1 - \text{i}$ is in the 4th quadrant so we need to subtract the angle from 2π (360^{o})

Arg $(1 - \text{i}) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$1 - \text{i} \equiv \left[\sqrt{2}, \frac{7\pi}{4}\right]$

2. (Review of last lesson) Express the complex number 6 cis $\frac{\pi}{3}$ in Cartesian form.

Working: 6 cis $\frac{\pi}{3} \equiv 6\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3} = 3 + 3\sqrt{3}\text{i}$

3. Let $z_1 = 5 + 4\text{i}$ and let $z_2 = 1 - \text{i}$.
   (a) Find $|z_1 - z_2|$.
   (b) State what $|z_1 - z_2|$ represents geometrically.

Working: (a) $|4 - 3\text{i}| = 5$

(b) The distance between z_1 and z_2.

E.g. 1 (a) Put the equation $|z| = 2$ into words.
(b) Hence describe the loci of all points such that $|z| = 2$.

Working: (a) The modulus (or magnitude) of the complex number z is 2.

(b) A circle of radius 2 whose centre is the origin.

E.g. 2 (a) Put the equation $|z - (4 + 3\text{i})| = 5$ into words.
(b) Hence, on an Argand diagram, draw all the points such that $|z - (4 + 3\text{i})| = 5$.

Working: (a) The distance between a complex number and $(4 + 3\text{i})$ is 5.

(b) A circle , centre (4, 3), radius 5.

Perpendicular bisectors

In words, $|z - z_1| = |z - z_2|$ is "the distance between z and the complex number z_1 is equal to the distance between z and the complex number z_2.

$|z - z_1| = |z - z_2|$ ***is the perpendicular bisector of the line segment joining z_1 and z_2.***

E.g. 3 (a) Put the equation $|z-3| = |z-5|$ into words.
(b) Hence, on an Argand diagram, draw all the points such that $|z-3| = |z-5|$.

Working: (a) $|z-3| = |z-5|$ is the distance between a complex number and 3 is equal to the distance between the same complex number and 5.

(b) Vertical line passing through Re 4
i.e. the perpendicular bisector of the line segment connecting the points 3 and 5

E.g. 4 Describe all the points such that arg $\arg(z-(5+2\text{i})) = \frac{\pi}{3}$.

Working: Half-line, from the point $5+2\text{i}$, at angle of $\frac{\pi}{3}$ to the positive real axis.
Is the point origin $5+2\text{i}$ included in the half-line? No

E.g. 5 When would the point z_1 be included in the half-line?

Working: When z_1 has the same argument as θ.

E.g. 6 Describe all the points such that:

(a) $|z-3+2\text{i}| = 9$

(b) $\arg\left(z-\left(1+2\text{i}\right)\right) = \frac{\pi}{6}$

(c) $\left|z-(4+3\text{i})\right| = \left|z+6-2\text{i}\right|$

(d) $\arg(z-1-\text{i}) = \frac{\pi}{4}$

Working: (a) $|z-3+2\text{i}| = 9 \quad \Rightarrow \quad \left|z-(3-2\text{i})\right| = 9$
Circle, centre 3 — 2i, radius 9

(b) Half-line starting at, but not including 1 + 2i,
which makes an angle of $\frac{\pi}{6}$ with the positive $x-$axis

(c) Rewrite as $\left|z-(4+3\text{i})\right| = \left|z-(-6+2\text{i})\right|$
Perpendicular bisector of the points $4+3\text{i}$ and $-6+2\text{i}$

(d) $\arg(z-1-\text{i}) = \arg\left(z-(1+\text{i})\right)$
Half-line starting at and including 1 + i,
which makes an angle of $\frac{\pi}{4}$ with the positive $x-$axis

E.g. 7 Use an Argand diagram to find, in the form $a+b\text{i}$, the complex number(s) which satisfies

(a) $\arg(z+1) = \frac{\pi}{4}$ and $\arg(z-3) = \frac{3\pi}{4}$

(b) $\arg z = \frac{\pi}{6}$ and $|z| = 2$.

N.B. The gradient of a straight line is equal to $\tan\theta$, where θ is the angle the line makes with the positive $x-$axis.

Hint: draw an Argand diagram.

Working: (a) $\arg(z+1)=\frac{\pi}{4}$ is the half-line starting at $-1+0\text{i}$ at an angle of $\frac{\pi}{4}$ (45°) – this is equivalent to the line $y=x+1$ for $x>-1$

$\arg(z-3)=\frac{3\pi}{4}$ is the half-line starting at $3+0\text{i}$ at an angle of $\frac{3\pi}{4}$ (135°) – this is equivalent to the line $y=3-x$ for $x<3$

Hence find the point of intersection of $y=x+1$ and $y=3-x$

The point is $(1,2)$ so $z=1+2\text{i}$

(b) $\arg z=\frac{\pi}{6}$ is the half-line starting at the origin at an angle of $\frac{\pi}{6}$ (30°) – this is equivalent to the line $y=\sqrt{3}x$ for $x>0$

N.B. The gradient of a line, m, is

$|z|=2$ is the circle, centre at the origin, radius 2 i.e. $x^2+y^2=4$

Hence find the point of intersection of $y=\frac{1}{\sqrt{3}}x$ and $x^2+y^2=4$

$x^2+\frac{1}{3}x^2=4 \quad\Rightarrow\quad \frac{4}{3}x^2=4 \quad\Rightarrow\quad x=\pm\sqrt{3}$

Since $x>0$, $x=\sqrt{3}$.

When $x=\sqrt{3}$, $y=1$

$\therefore z=\sqrt{3}+\text{i}$

E.g. 8 If $\arg z=\frac{\pi}{4}$ and $\arg(z-3)=\frac{\pi}{2}$, find $\arg(z-6\text{i})$.

Hint: use an Argand diagram.

Working: $\arg z=\frac{\pi}{4}$ is the half-line starting at the origin at an angle of $\frac{\pi}{4}$ (45°) – this is equivalent to the line $y=x$ for $x>0$

$\arg(z-3)=\frac{\pi}{2}$ is the half-line starting at $3+0\text{i}$ at an angle of $\frac{\pi}{2}$ (90°) – this is equivalent to the vertical line $x=3$ for $y>0$

The lines meet at the point $3+3\text{i}$ so $z=3+3\text{i}$

$\arg(z-6\text{i})=\arg(3-3\text{i})$

Ignore the signs: $\tan^{-1}\frac{3}{3}=\frac{\pi}{4}$

$3-3\text{i}$ is in the 4th quadrant so we need to subtract the angle from 2π (360°)

$\text{Arg}(3-3\text{i})=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$

Or

The line from 6i to $3+3\text{i}$ has to turn clockwise through $\frac{\pi}{4}$ to get to $3+3\text{i}$

$\arg(z-6\text{i})=-\frac{\pi}{4}$

Video: Loci of a circle
Video: Loci of perpendicular bisectors
Video: Loci of half-lines
Video: Representing regions in Argand diagrams

Solutions to Starter and E.g.s

Exercise
p133 4F Qu 1i, 2i, 3i, 4i, 5-11