Operations in Modulus-Argument Form

Starter

1. **(Review of last lesson)** Use an Argand diagram to find, in the form $a + b\text{i}$, the complex number(s) which satisfies $\arg(z - 4\text{i}) = \pi$ and $|z + 6| = 5$.

Working: $\arg(z - 4\text{i}) = \pi$ is the half-line starting at the point 4i at an angle of π (180^{o}) – this is equivalent to the line $y = 4$ for $x < 0$
$|z + 6| = 5$ is a circle of radius 5, centre $(-6, 0)$ – this has equation
$(x+6)^2 + y^2 = 25 \quad \Rightarrow \quad (x+6)^2 + 4^2 = 25 \quad \Rightarrow \quad x + 6 = \pm 3$
$x = -3$ or $x = -9$
So $z = -3 + 4\text{i}$ or $z = -9 + 4\text{i}$

2. Let $z_1 = r_1(\cos\theta_1 + \text{i}\ \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)$
State the values of:

(a) $|z_1|$ (b) $|z_2|$ (c) $|z_1 \times z_2|$ (d) $\left|\dfrac{z_1}{z_2}\right|$

Working: (a) r_1 (b) r_2 (c) $r_1 \times r_2$ (d) $\dfrac{r_1}{r_2}$

E.g. 1 Let $z_1 = r_1(\cos\theta_1 + \text{i}\ \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)$.

(a) State the values of: (i) $\arg z_1$ (ii) $\arg z_2$

(b) Find the values of: (i) $\arg(z_1 \times z_2)$ (ii) $\arg\left(\dfrac{z_1}{z_2}\right)$

Working: (a) (i) θ_1
(ii) θ_2

(b) (i) $z_1 \times z_2 = r_1(\cos\theta_1 + \text{i}\ \sin\theta_1) \times r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)$

$$= r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \text{i}\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right)$$

$$= r_1 r_2\left(\cos(\theta_1 + \theta_2) + \text{i}\,(\sin(\theta_1 + \theta_2)\right)$$

So $\arg(z_1 \times z_2) = \theta_1 + \theta_2$

(ii) To find the arg $\left(\dfrac{z_1}{z_2}\right)$ we must multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \text{i}\ \sin\theta_1)}{r_2(\cos\theta_2 + \text{i}\ \sin\theta_2)} \times \frac{r_2(\cos\theta_2 - \text{i}\ \sin\theta_2)}{r_2(\cos\theta_2 - \text{i}\ \sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + \text{i}\,(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + \text{i}\,(\sin(\theta_1 - \theta_2)\right)$$

So $\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2$

E.g. 2 Let $z_1 = -1 + \sqrt{3}\text{i}$ and $z_2 = \sqrt{3} + \text{i}$. By giving your answers $0 \leq \theta < 2\pi$, find:

(a) $|z_1 \times z_2|$ and Arg $(z_1 \times z_2)$

(b) $|z_1 \div z_2|$ and Arg $(z_1 \div z_2)$

(c) Arg (z_1^4)

Working:

(a) $|z_1| = |-1 + \sqrt{3}\text{i}| = \sqrt{(-1)^2 + 3} = 2$

$|z_2| = |\sqrt{3} + \text{i}| = 2$

So $|z_1 \times z_2| = 4$

Arg z_1 = Arg $(-1 + \sqrt{3}\text{i}) = \dfrac{2\pi}{3}$

Arg z_2 = Arg $(\sqrt{3} + \text{i}) = \dfrac{\pi}{6}$

Arg $(z_1 \times z_2)$ = Arg z_1 + Arg $z_2 = \dfrac{2\pi}{3} + \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

(b) $|z_1 \div z_2| = \dfrac{|z_1|}{|z_2|} = 1,$

Arg $(z_1 \div z_2)$ = Arg z_1 − Arg $z_2 = \dfrac{2\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{2}$

(c) Arg $(z_1^4) = 4 \times$ Arg $z_1 = 4 \times \dfrac{2\pi}{3} = \dfrac{8\pi}{3}$

$\dfrac{8\pi}{3}$ is outside the range $0 \leq \theta < 2\pi$ so subtract 2π

$\dfrac{8\pi}{3} - 2\pi = \dfrac{2\pi}{3}$

Arg $(z_1^4) = \dfrac{2\pi}{3}$

Video: Operations in modulus-argument form

Complex numbers EQ

Solutions to Starter and E.g.s

Exercise

p137 4G Qu 1i, 2i, 3-5 (6 requires trigonometric identities)