Scalar product and angles

Starter

1. Find the position vector of the point of intersection of $\frac{x+2}{4} = y = \frac{z-1}{2}$ and $\frac{x}{2} = \frac{y+1}{2} = 4 - z$.

Working: Equating components:

$\mathbf{i}$: $-2 + 4s = 2t \quad \Rightarrow \quad 2s - t = 1$

$\mathbf{j}$: $s = -1 + 2t \quad \Rightarrow \quad s - 2t = -1$

$\mathbf{k}$: $1 + 2s = 4 - t \quad \Rightarrow \quad 2s + t = 3$

Solving the $\mathbf{i}$ and $\mathbf{j}$ equations simultaneously: $s = 1, t = 1$

Check values work in the $\mathbf{k}$ equation: $1 + 2 = 4 - 1$ ✓

Substitute either the $s-$ or the $t-$value in the relevant equation:

Either… $s = 1$ $\quad \mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

…or… $t = 1$: $\quad \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

The point of intersection between the lines is $(2, 1, 3)$.

2. Find the magnitude (modulus) of the following vectors:

(a) $\begin{pmatrix} -6 \\ 5 \end{pmatrix}$ (b) $5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Working: (a) $\left| \begin{pmatrix} -6 \\ 5 \end{pmatrix} \right| = \sqrt{(-6)^2 + 5^2} = \sqrt{61}$

(b) $\left| 5\mathbf{i} - 3\mathbf{j} + \mathbf{k} \right| = \sqrt{5^2 + (-3)^2 + 1^2} = \sqrt{35}$

E.g. 1 (a) Using the definition of the scalar product, $\mathbf{a} \,.\, \mathbf{b} = |a||b|\cos\theta$, write down the values of: (i) $\mathbf{i} \,.\, \mathbf{i}$ (ii) $\mathbf{i} \,.\, \mathbf{j}$

(b) Hence write down the values of:

(i) $\mathbf{j} \,.\, \mathbf{j}$ (ii) $\mathbf{k} \,.\, \mathbf{k}$ (iii) $\mathbf{i} \,.\, \mathbf{k}$ (iv) $\mathbf{j} \,.\, \mathbf{k}$

(c) Find the value of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

Working: (a) (i) $\mathbf{i} \,.\, \mathbf{i} = |\mathbf{i}||\mathbf{i}| \times \cos 0 = 1 \times 1 \times \cos 0 = 1$

(ii) $\mathbf{i} \,.\, \mathbf{j} = |\mathbf{i}||\mathbf{j}| \times \cos 90 = 1 \times 1 \times \cos 90 = 0$

(b) (i)/(ii) $\mathbf{j} \,.\, \mathbf{j} = \mathbf{k} \,.\, \mathbf{k} = 1$

(iii)/(iv) $\mathbf{i} \,.\, \mathbf{k} = \mathbf{j} \,.\, \mathbf{k} = 0$

(c) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$

E.g. 2 Find the scalar products of the following pairs of vectors:

(a) $\begin{pmatrix}4\\6\end{pmatrix} . \begin{pmatrix}3\\10\end{pmatrix}$ (b) $(3\mathbf{i}+5\mathbf{j}) . (8\mathbf{i}-5\mathbf{j})$

Working: (a) $\begin{pmatrix}4\\6\end{pmatrix} . \begin{pmatrix}3\\10\end{pmatrix} = 3\times 4 + 6\times 10 = 72$

(b) $(3\mathbf{i}+5\mathbf{j}) . (8\mathbf{i}-5\mathbf{j}) = 3\times 8 + 5\times(-5) = -1$

E.g. 3 Given that $\mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, find the value of $\mathbf{v} . \mathbf{v}$ in terms of $\mathbf{v}$?

Working: $\mathbf{v} . \mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} . \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1^2 + v_2^2 + v_3^2$

But $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

So $\mathbf{v} . \mathbf{v} = |\mathbf{v}|^2$

E.g. 4 Using the scalar product find the angle between the two vectors $\begin{pmatrix}5\\7\end{pmatrix}$ and $\begin{pmatrix}6\\-2\end{pmatrix}$.

Working: $\begin{pmatrix}5\\7\end{pmatrix} . \begin{pmatrix}6\\-2\end{pmatrix} = 5\times 6 + 7\times(-2) = 16$

$\left|\begin{pmatrix}5\\7\end{pmatrix}\right| = \sqrt{5^2+7^2} = \sqrt{74}$ and $\left|\begin{pmatrix}6\\-2\end{pmatrix}\right| = \sqrt{6^2+(-2)^2} = 2\sqrt{10}$

Using $\cos\theta = \dfrac{\mathbf{a} . \mathbf{b}}{|a||b|}$: $\cos\theta = \dfrac{16}{\sqrt{74}\times 2\sqrt{10}}$

The angle between the vectors is $72.9°$.

E.g. 5 For each of the following pairs of vectors find the angle between them:

(a) $3\mathbf{i}-\mathbf{j}$ and $5\mathbf{i}-2\mathbf{j}$ (b) $\begin{pmatrix}-2\\8\end{pmatrix}$ and $\begin{pmatrix}-5\\-9\end{pmatrix}$

Working: (a) $(3\mathbf{i}-\mathbf{j}) . (5\mathbf{i}-2\mathbf{j}) = 3\times 5 + (-1)\times(-2) = 17$

$|3\mathbf{i}-\mathbf{j}| = \sqrt{3^2+(-1)^2} = \sqrt{10}$

$|5\mathbf{i}-2\mathbf{j}| = \sqrt{5^2+(-2)^2} = \sqrt{29}$

Using $\cos\theta = \dfrac{\mathbf{a} . \mathbf{b}}{|a||b|}$: $\cos\theta = \dfrac{17}{\sqrt{10}\times 2\sqrt{29}}$

The angle between the vectors is $3.6°$.

(b) $\begin{pmatrix}-2\\8\end{pmatrix}.\begin{pmatrix}-5\\-9\end{pmatrix}=(-2)\times(-5)+8\times(-9)=-62$

$\left|\begin{pmatrix}-2\\8\end{pmatrix}\right|=\sqrt{(-2)^2+8^2}=\sqrt{68}=2\sqrt{17}$

$\left|\begin{pmatrix}-5\\-9\end{pmatrix}\right|=\sqrt{(-5)^2+(-9)^2}=\sqrt{106}$

Using $\cos\theta=\dfrac{\mathbf{a}.\mathbf{b}}{|a||b|}$: $\cos\theta=\dfrac{-62}{2\sqrt{17}\times\sqrt{106}}$

The angle between the vectors is $136.9°$.

E.g. 6 What can be said about the angle between the two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ if:
(a) $\mathbf{a}.\mathbf{b}=0$ (b) $\mathbf{a}.\mathbf{b}>0$ (b) $\mathbf{a}.\mathbf{b}<0$

Working: (a) $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors $\Rightarrow$ $|\mathbf{a}|\neq 0$ and $|\mathbf{b}|\neq 0$
So if $\mathbf{a}.\mathbf{b}=0$ $\Rightarrow$ $\cos\theta=0$ $\Rightarrow$ $\theta=90°$
Therefore, the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to one another.

(b) Since $|\mathbf{a}|>0$ and $|\mathbf{b}|>0$, if $\mathbf{a}.\mathbf{b}>0$ $\Rightarrow$ $\cos\theta>0$
Therefore, the angle between the vectors is acute.

(c) Since $|\mathbf{a}|>0$ and $|\mathbf{b}|>0$, if $\mathbf{a}.\mathbf{b}<0$ $\Rightarrow$ $\cos\theta<0$
Therefore, the angle between the vectors is obtuse.

E.g. 7 Find the acute angle between the lines $\mathbf{r}=\begin{pmatrix}2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}7\\-2\end{pmatrix}+\mu\begin{pmatrix}-3\\5\end{pmatrix}$

Working: Since each line is parallel to its direction vector, we can find the angle between the lines by finding the angle between the direction vectors.

$\begin{pmatrix}1\\2\end{pmatrix}.\begin{pmatrix}-3\\5\end{pmatrix}=1\times(-3)+2\times5=7$

$\left|\begin{pmatrix}1\\2\end{pmatrix}\right|=\sqrt{1^2+2^2}=\sqrt{5}$

$\left|\begin{pmatrix}-5\\-9\end{pmatrix}\right|=\sqrt{(-3)^2+5^2}=\sqrt{34}$

Using $\cos\theta=\dfrac{\mathbf{a}.\mathbf{b}}{|a||b|}$: $\cos\theta=\dfrac{7}{\sqrt{5}\times\sqrt{34}}$

The angle between the lines is $57.5°$.

E.g. 8 Find the angles between the following pairs of lines:

(a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \mu(7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$

(b) $\mathbf{r} = (2 + 6\lambda)\mathbf{i} + (1 - 3\lambda)\mathbf{j} + 2\lambda\mathbf{k}$ and $\mathbf{r} = (2 + 4\mu)\mathbf{i} + 7\mathbf{j} + (5 + 3\mu)\mathbf{k}$

Working:

(a) Find the angle between the vectors $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

$(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \,.\, (7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) = 14 - 4 + 8 = 18$

$\left|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}\right| = \sqrt{4 + 1 + 4} = 3$

$\left|7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}\right| = \sqrt{49 + 16 + 16} = 9$

Using $\cos\theta = \dfrac{\mathbf{a} \,.\, \mathbf{b}}{|a||b|}$***:*** $\cos\theta = \dfrac{18}{3 \times 9}$

The angle between the lines is $48.2°$.

(b) Find the angle between the vectors $6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{k}$

$(6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \,.\, (4\mathbf{i} + 3\mathbf{k}) = 24 + 6 = 30$

$\left|6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}\right| = \sqrt{36 + 9 + 4} = 7$

$\left|4\mathbf{i} + 3\mathbf{k}\right| = \sqrt{16 + 9} = 5$

Using $\cos\theta = \dfrac{\mathbf{a} \,.\, \mathbf{b}}{|a||b|}$***:*** $\cos\theta = \dfrac{30}{7 \times 5}$

The angle between the lines is $31.0°$.

Video: Scalar product

Video: Angle between 2 lines

Scalar product EQ

Solutions to Starter and E.g.s

Exercise

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