Sum and Product of Roots (Quadratics)

Starter

1. **(Review of last lesson)**
The equation $z^4 - 4z^3 + 3z^2 + 2z - 6 = 0$ has a root $1 - \mathrm{i}$. Find the other roots.

Working: If $1 - \mathrm{i}$ is a root, then so is $1 + \mathrm{i}$.
$(z - (1 - \mathrm{i}))(z - (1 + \mathrm{i})) = z^2 - 2z + 2$
$\therefore z^4 - 4z^3 + 3z^2 + 2z - 6 = (az^2 + bz + c)(z^2 - 2z + 2)$
By inspection, $a = 1$ and $c = -3$
Equating coefficients: 'z': $2 = 2b - 2c$
Since $c = -3$ $b = -2$
$\therefore z^4 - 4z^3 + 3z^2 + 2z - 6 = (z^2 - 2z - 3)(z^2 - 2z + 2)$
$z^2 + 4z - 3 = (z - 3)(z - 1)$
The other roots are 3, -1 and $1 + \mathrm{i}$

2. The quadratic formula gives the two solutions of the equation $ax^2 + bx + c = 0$ as $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
Find: (a) the sum of the roots
(b) the product of the roots.

Working: (a) $$\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = = \frac{-b}{a}$$
$\therefore$ sum of roots = $-\dfrac{b}{a}$

(b) $$\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$
$\therefore$ product of roots = $\dfrac{c}{a}$

3. Using your answers to question 2, ***write down*** the sum and product of the roots of the quadratic equation $12x^2 + 11x - 56 = 0$.

Working: sum of roots = $-\dfrac{11}{12}$ (b) product of roots = $-\dfrac{56}{12} = -\dfrac{14}{3}$

E.g. 1 Find the simplest quadratic equation with the roots 2 and 3.

Working: The simplest cubic has 1 as the coefficient of x^3.
Sum of roots $= 5$ ***(change the sign)***
Product of roots $= 6$ ***(don't change the sign)***
So simplest equation is $x^2 - 5x + 6 = 0$.

E.g. 2 Write down a quadratic equation with integer coefficients such that the sum of the roots is $\frac{1}{2}$ and the product is 3.

Working: The simplest equation is $x^2 - \frac{1}{2}x + 3 = 0$.

To get integer coefficients multiply by 2: $2x^2 - x + 6 = 0$

E.g. 3 Write down the expansions of: (a) $(\alpha + \beta)^2$ (b) $(\alpha - \beta)^2$

Working: (a) $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

(b) $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

E.g. 4 Let the roots of the equation $x^2 - 4x + 2 = 0$ be α and β. Without finding α and β, find the values of the following:

(a) $3\alpha + 3\beta$ (b) $\alpha^2 + 2\alpha\beta + \beta^2$ (c) $\alpha^2 - \alpha\beta + \beta^2$

(d) $(\alpha - \beta)^2$ (e) $\frac{1}{\alpha} + \frac{1}{\beta}$ (f) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Hint: find $\alpha + \beta$ and $\alpha\beta$ then express each question in terms of them.
For (e) and (f), add the algebraic fractions.

Working: (a) Sum of roots $= \alpha + \beta = 4$

Product of roots $= \alpha\beta = 2$

$3\alpha + 3\beta = 3(\alpha + \beta) = 3 \times 4 = 12$

(b) $\alpha^2 + 2\alpha\beta + \beta^2 = (\alpha + \beta)^2 = 4^2 = 16$

(c)
$$\begin{aligned}\alpha^2 - \alpha\beta + \beta^2 &= \alpha^2 + \beta^2 - \alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta - \alpha\beta \\ &= 4^2 - 3 \times 2 \\ &= 10\end{aligned}$$

(d)
$$\begin{aligned}(\alpha - \beta)^2 &= \alpha^2 + \beta^2 - 2\alpha\beta \\ &= \alpha^2 - \alpha\beta + \beta^2 - \alpha\beta \\ &= 10 - 2 \\ &= 8\end{aligned}$$

this now looks similar to (c)

(e) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{4}{2} = 2$

(f)
$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2}{\alpha^2\beta^2}+\frac{\alpha^2}{\alpha^2\beta^2}$$
$$=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$$
$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$
$$=\frac{4^2-2\times 2}{2^2}$$
$$=3$$

Video: Sum and product of roots (quadratic)

Solutions to Starter and E.g.s

Exercise
p150 5C Qu 1, 2