Transforming equations

Starter

1. **(Review of last lesson)**

The roots of the equation $ax^2 + bx + c = 0$ differ by 1. Prove that $b^2 - a^2 - 4ac = 0$.

Working: Let the roots be α and $\alpha + 1$.

Sum of roots is $2\alpha + 1 = -\frac{b}{a} \quad \Rightarrow \quad \alpha = -\frac{b}{2a} - \frac{1}{2}$

Product of roots is $\alpha(\alpha + 1) = \frac{c}{a} \quad \Rightarrow$

$$\left(-\frac{b}{2a} - \frac{1}{2}\right)\left(-\frac{b}{2a} - \frac{1}{2} + 1\right) = \frac{c}{a}$$

Multiply by $(-4a)$: $\quad (b + a)(-b - a + 2a) = -4ac$

$$(a + b)(a - b) = -4ac$$

$$a^2 - b^2 = -4ac$$

$$b^2 - a^2 - 4ac = 0 \qquad \text{QED}$$

N.B. QED is short for "quod erat demonstrandum" and is Latin for "which was to be demonstrated/proved".

E.g. The quadratic equation $x^2 + 5x + 7 = 0$ has roots α and β. Without calculating α and β, find an equation with roots 2α and 2β.

Working: Let u be one of the new roots so $u = 2\alpha \quad \Rightarrow \quad \alpha = \frac{1}{2}u$.

Since α satisfies $x^2 + 5x + 7 = 0$ then $\left(\frac{1}{2}u\right)^2 + 5\left(\frac{1}{2}u\right) + 7 = 0$

Simplifying: $\quad \frac{1}{4}u^2 + \frac{5}{2}u + 7 = 0$

$\therefore$ new equation is $\quad u^2 + 10u + 28 = 0$

Since this is true if u is either 2α or 2β, $u^2 + 10u + 28 = 0$ is the required equation.

E.g. 1 The equation $x^2 + 4x + 7 = 0$ has roots α and β. Use the "substitution method" to find equations with integer coefficients which have the following roots.

(a) 3α and 3β (b) α^2 and β^2 (c) $\alpha + 2\beta$ and $2\alpha + \beta$

Working: (a) Let u be one of the new roots so $u = 3\alpha \quad \Rightarrow \quad \alpha = \frac{1}{3}u$.

Since α satisfies $x^2 + 4x + 7 = 0$ then

$$\left(\frac{1}{3}u\right)^2 + 4\left(\frac{1}{3}u\right) + 7 = 0$$

Simplifying: $\quad \frac{1}{9}u^2 + \frac{4}{3}u + 7 = 0$

$\therefore$ new equation is $\quad u^2 + 12u + 63 = 0$

Since this is true if u is either 3α or 3β, $u^2 + 12u + 63 = 0$ is the required equation.

(b) Let $u = \alpha^2$ then $\alpha = \pm\sqrt{u}$

Substituting gives: $u \pm 4\sqrt{u} + 7 = 0$

Rearranging: $\pm 4\sqrt{u} = -u - 7$

Squaring both sides gets rid of $\pm$: $16u = u^2 + 14u + 49$

$u^2 - 2u + 49 = 0$

If $u = \beta^2$, we would get the same result so

$u^2 - 2u + 49 = 0$ is the required equation

(c) Let $u = \alpha + 2\beta$ so $\alpha = u - 2\beta$

$(u - 2\beta)^2 + 4(u - 2\beta) + 7 = 0$

Expand: $u^2 + 4(1 - \beta)u + 4\beta(\beta - 2) + 7 = 0$

What is β? This method doesn't seem to work so let's try…

If $u = \alpha + 2\beta$ then:

$u + \alpha = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times -4 = -8$

Remember: $\alpha + \beta = -\frac{b}{a} = -4$

So $\alpha = -u - 8$

Substitute: $(-u - 8)^2 + 4(-u - 8) + 7 = 0$

$u^2 + 12u + 39 = 0$

E.g. 2 The equation $2x^3 + 3x^2 + 4x + 5 = 0$ has roots α, β and γ. Find the equation whose roots are α^{-1}, β^{-1} and γ^{-1}.

Working: Let u be one of the new roots so $u = \alpha^{-1} \Rightarrow \alpha = u^{-1}$

$2(u^{-1})^3 + 3(u^{-1})^2 + 4u^{-1} + 5 = 0$

Multiplying by u^3 gives: $5u^3 + 4u^2 + 3u + 2 = 0$

E.g. 3 The equation $x^3 + 3x^2 + 4x + 5 = 0$ has roots α, β and γ. Find the equation whose roots are $\beta + \gamma$, $\gamma + \alpha$ and $\alpha + \beta$.

Hint: Similar to E.g. 1(c)

Working: Let $u = \beta + \gamma$.

Then $u + \alpha = \alpha + \beta + \gamma = -3$ so $\alpha = -u - 3$

$(-u - 3)^3 + 3(-u - 3)^2 + 4(-u - 3) + 5 = 0$

Expand and simplify: $u^3 + 6u^2 + 13u + 7 = 0$

If $u = \gamma + \alpha$ or $u = \alpha + \beta$ the same result would be found so

$u^3 + 6u^2 + 13u + 7 = 0$ is the required equation.

E.g. 4 The equation $x^3 - 9x^2 + 31x - 39 = 0$ has roots α, β and γ which are in arithmetic progression. Solve the equation.

N.B. From GCSE, arithmetic progression means that terms are found by adding the same amount each time e.g. $a,\ a + d,\ a + 2d, \ldots$.

Hint: Rather than α, $\alpha + d$ and $\alpha + 2d$ let the roots be $\alpha - d$, α and $\alpha + d$

Working: Let the roots be $\alpha - d$, α and $\alpha + d$.

The sum of the roots is $3\alpha \quad \Rightarrow \quad 3\alpha = 9 \quad \Rightarrow \quad \alpha = 3$

If $x = 3$ is a root then $x - 3$ is a factor.

Factorising: $x^3 - 9x^2 + 31x - 39 \equiv (x-3)(x^2 - 6x + 13)$

Solving $x^2 - 6x + 13 = 0$ gives $x = 3 \pm 2\text{i}$

$\therefore$ the roots are 3 and $x = 3 \pm 2\text{i}$

E.g. 5 The roots of the equation $x^3 + ax^2 + bx + c = 0$, where $c \neq 0$ are in geometric progression. Prove that $b^3 = a^3c$.

N.B. From GCSE, geometric progression means that terms are found by multiplying by the same number each time e.g. $a,\ ar,\ ar^2, \ldots$.

Hint: Rather than α, αr and αr^2 let the roots be $\frac{\alpha}{r}$, α and αr

Working: Let the roots be $\frac{\alpha}{r}$, α and $r\alpha$.

The product of the roots is $\alpha^3 = -c \quad \Rightarrow \quad \alpha = \sqrt[3]{-c}$

Given that α satisfies the cubic: $(\sqrt[3]{-c})^3 + a(\sqrt[3]{-c})^2 + b(\sqrt[3]{-c}) + c = 0$

$\therefore -c + ac^{\frac{2}{3}} - bc^{\frac{1}{3}} + c = 0 \quad \Rightarrow \quad ac^{\frac{1}{3}} = b$

Cubing both sides gives the required result: $b^3 = a^3c$

Video: Transforming equations

Solutions to Starter and E.g.s

Exercise

p162 5F Qu 1i, 2-8, (9-12 red)