Vector equation of a line

Starter

1. A 3-D transformation is defined as a stretch by a factor of 2 in the $x-$direction, followed by a rotation of $45°$ about the $z-$axis. Find the matrix which defined this mapping.

Working: Stretch by a factor of 2 in the $x-$direction: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\theta°$ about the $z-$axis is $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$45°$ about the $z-$axis is $\begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

$$\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

E.g. 1 Find the vector equation of the line passing through $A(3, 2, 1)$ and $B(1, 4, 2)$.

Working: The equation is of the form $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$

To find $\mathbf{d}$ we can find $\overrightarrow{AB}$ or $\overrightarrow{BA}$ and $\mathbf{p}$ can be either point

Either $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ or $\mathbf{p} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

Either $\mathbf{d} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

…or… $\mathbf{d} = \overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

The equation of the line could be $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$.

N.B. Other combinations of the vectors above are also possible.

E.g. 2 Find the equation of the line that passes through the points $(7, 2)$ and $(-5, 6)$.

Working: $\mathbf{p} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ or $\mathbf{p} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$

$\mathbf{d} = \begin{pmatrix} -5 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 4 \end{pmatrix}$ or $\mathbf{d} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \end{pmatrix}$

Multiples of $\mathbf{d}$ are also parallel to the line so better is $\mathbf{d} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

The equation of the line is $\mathbf{r} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

N.B. Other answers also possible.

Showing a point lies on the line

E.g. 2 Decide whether the point $P(9, 5, 8)$ lies on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$.

Working: Put the line equal to the point: $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \begin{vmatrix} 9 \\ 5 \\ 8 \end{vmatrix}$

Equating components:

$\mathbf{i}$: $1 + 4\lambda = 9 \Rightarrow \lambda = 2$

$\mathbf{j}$: $3 + \lambda = 5 \Rightarrow \lambda = 2$

$\mathbf{k}$: $2 + 3\lambda = 8 \Rightarrow \lambda = 2$

Since the $\lambda-$values are all the same, the point lies on the line.

E.g. 3 Decide whether $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{vmatrix} 9 \\ 0 \\ 8 \end{vmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ are the same line or not.

Working: $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ is a multiple of $\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ so the lines are at least parallel.

Does the point $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ lie on the line $\mathbf{r} = \begin{vmatrix} 9 \\ 0 \\ 8 \end{vmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$?

$$\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{vmatrix} 9 \\ 0 \\ 8 \end{vmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

Equating components:

$\mathbf{i}$: $5 = 9 - 2\mu \Rightarrow \mu = 2$

$\mathbf{j}$: $2 = \mu$

$\mathbf{k}$: $1 = 8 - 3\mu \Rightarrow \mu = \frac{7}{3}$

Since the $\mu-$values are not all the same, the fixed point of one line does not lie on the other line. Therefore, they are not the same line.

E.g. 4 Decide whether $\mathbf{r} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -8 \\ 4 \end{pmatrix}$ are the same line or not.

Working: $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ is a multiple of $\begin{pmatrix} -6 \\ -8 \\ 4 \end{pmatrix}$ so the lines are at least parallel.

Does the point $\begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$ lie on the line $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -8 \\ 4 \end{pmatrix}$?

$$\begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -8 \\ 4 \end{pmatrix}$$

Equating components:

$$\begin{aligned} \mathbf{i}: &\quad -3 = 3 - 6\mu \quad \Rightarrow \quad \mu = 1 \\ \mathbf{j}: &\quad -3 = 5 - 8\mu \quad \Rightarrow \quad \mu = 1 \\ \mathbf{k}: &\quad 3 = -1 + 4\mu \quad \Rightarrow \quad \mu = 1 \end{aligned}$$

Since the $\mu-$values are all the same, the fixed point of one line lies on the other line. Since they are also parallel, they are the same line.

Video: Vector equation of lines
Video: Parallel lines

Solutions to Starter and E.g.s

Exercise
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