The vector product

Starter

1. Find the angle between lines $\mathbf{r} = (5 - 2\lambda)\mathbf{i} + 4\lambda\mathbf{j}$ and $\mathbf{r} = (5 + \mu)\mathbf{i} + (3 - 7\mu)\mathbf{j}$.

Working: Since each line is parallel to its direction vector, we can find the angle between the lines by finding the angle between the direction vectors.

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ -7 \end{pmatrix} = -2 - 28 = -30$$

$$\left| \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\left| \begin{pmatrix} 1 \\ -7 \end{pmatrix} \right| = \sqrt{1 + 49} = \sqrt{50} = 5\sqrt{2}$$

Using $\cos\theta = \dfrac{\mathbf{a} . \mathbf{b}}{|a||b|}$: $\quad \cos\theta = \dfrac{-30}{2\sqrt{5} \times 5\sqrt{2}}$

The angle between the lines is $161.6°$.

2. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, simplify:

(a) $(\mathbf{a} - \mathbf{b}) . (\mathbf{a} - \mathbf{b})$

(b) $(\mathbf{a} - \mathbf{b}) . \mathbf{b} - (\mathbf{b} - \mathbf{a}) . \mathbf{a}$

(c) $(2\mathbf{a} + 3\mathbf{b}) . \mathbf{b}$

Working:

(a) Since $\mathbf{a}$ and $\mathbf{b}$: $\quad \mathbf{a} . \mathbf{b} = 0$

In general: $\quad \mathbf{a} . \mathbf{a} = |\mathbf{a}|^2 \quad$ and $\quad \mathbf{b} . \mathbf{b} = |\mathbf{b}|^2$

$$\begin{aligned}(\mathbf{a} - \mathbf{b}) . (\mathbf{a} - \mathbf{b}) &= \mathbf{a} . \mathbf{a} - \mathbf{a} . \mathbf{b} - \mathbf{b} . \mathbf{a} + \mathbf{b} . \mathbf{b} \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2\end{aligned}$$

(b)
$$\begin{aligned}(\mathbf{a} - \mathbf{b}) . \mathbf{b} - (\mathbf{b} - \mathbf{a}) . \mathbf{a} &= \mathbf{a} . \mathbf{b} - \mathbf{b} . \mathbf{b} - \mathbf{b} . \mathbf{a} + \mathbf{a} . \mathbf{a} \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2\end{aligned}$$

(c)
$$(2\mathbf{a} + 3\mathbf{b}) . \mathbf{b} = 2\mathbf{a} . \mathbf{b} + 3\mathbf{b} . \mathbf{b} = 3|\mathbf{b}|^2$$

3. The vector $\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 5 \\ 6 \end{pmatrix}$. Find α and β.

Working: Two vectors are perpendicular to each other if the scalar product is zero.

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} . \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = 4 + 2\alpha + 5\beta = 0 \quad \Rightarrow \quad 2\alpha + 5\beta = -4$$

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \begin{pmatrix} -3 \\ 5 \\ 6 \end{pmatrix} = -3 + 5\alpha + 6\beta = 0 \quad \Rightarrow \quad 5\alpha + 6\beta = -3$$

Solving simultaneously: $\quad \alpha = 3 \quad$ and $\quad \beta = -2$

E.g. 1 Using the definition of the vector product and the right-hand rule state the values of:

(a) (i) $\mathbf{i} \times \mathbf{i}$ (ii) $\mathbf{j} \times \mathbf{j}$ (iii) $\mathbf{k} \times \mathbf{k}$

(b) (i) $\mathbf{i} \times \mathbf{j}$ (ii) $\mathbf{j} \times \mathbf{i}$ (iii) $\mathbf{i} \times \mathbf{k}$

(iv) $\mathbf{k} \times \mathbf{i}$ (v) $\mathbf{j} \times \mathbf{k}$ (vi) $\mathbf{k} \times \mathbf{j}$

Remember: $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Working: (a) (i) $\mathbf{i} \times \mathbf{i} = |\mathbf{i}||\mathbf{i}|\sin 0 \times \hat{\mathbf{n}} = 0$

Using the same thinking for (ii) and (iii).

(ii) Similarly, $\mathbf{j} \times \mathbf{j} = 0$

(iii) $\mathbf{k} \times \mathbf{k} = 0$

(b) (i) $\mathbf{i} \times \mathbf{j} = |\mathbf{i}||\mathbf{k}|\sin 90 \times \hat{\mathbf{n}} = 1 \times 1 \times 1 \times \hat{\mathbf{n}} = \hat{\mathbf{n}}$

Using the right-hand rule where $\mathbf{i}$ is the thumb and $\mathbf{j}$ is the index finger, it means that the resultant vector is in the direction of the middle finger i.e. the $\mathbf{k}-$direction.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$

Using the same thinking for (ii)—(vi).

(ii) $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

(iii) $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

(iv) $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

(v) $\mathbf{j} \times \mathbf{k} = \mathbf{i}$

(vi) $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

E.g. 3 Find $\mathbf{p} \times \mathbf{q}$ when $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

Working: $\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -1 & 7 \end{vmatrix} = 17\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}$

E.g. 4 Find $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Working: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 5 & 4 & -1 \end{vmatrix} = -7\mathbf{i} + 7\mathbf{j} - 7\mathbf{k}$

E.g. 5 Write down the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$?

Working: Thinking simply: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ but $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ so $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

Alternatively: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and $\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$. When two adjacent rows are swapped over the sign of the determinant changes so $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

E.g. 6 What does $\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{b})$ equal? Hence write down the value of $\mathbf{b} \,.\, (\mathbf{a} \times \mathbf{b})$.

Working: Since $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{b}) = 0$

Similarly, $\mathbf{b} \,.\, (\mathbf{a} \times \mathbf{b}) = 0$

E.g. 7 Find a unit vector which is perpendicular to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Working:

$$(\mathbf{i} + \mathbf{j} + \mathbf{k}) \,.\, (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$|\,\mathbf{i} - 2\mathbf{j} + \mathbf{k}\,| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

The required vector is $\dfrac{\sqrt{6}}{6}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

E.g. 8 Given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = p\mathbf{j} + q\mathbf{k}$ and $\mathbf{a} \times \mathbf{b} = 2\mathbf{j} + \lambda\mathbf{k}$, find the values of the scalars p, q and λ.

Working:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 0 & p & q \end{vmatrix} = (2q + 2p)\mathbf{i} - q\mathbf{j} + p\mathbf{k} \equiv 2\mathbf{j} + \lambda\mathbf{k}$$

Equating components:

$\mathbf{j}$: $\quad -q = 2 \quad \Rightarrow \quad q = -2$

$\mathbf{i}$: $\quad 2q + 2p = 0 \quad \Rightarrow \quad p = 2$

$\mathbf{k}$: $\quad p = \lambda \quad \Rightarrow \quad \lambda = 2$

Video: Vector product

Vectors EQ

Solutions to Starter and E.g.s

Exercise
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