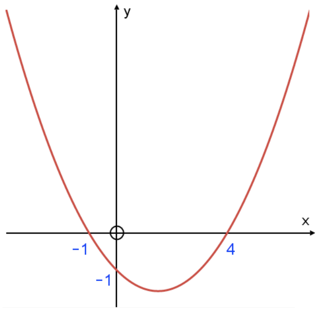
Completing the Square

Starter

1. **(Review of last lesson)**
The sketch is a quadratic function of the form $y = ax^2 + bx + x$. Find the value of a, b and c.



2. **(Review of GCSE material)**
(a) Express $x^2 - 2x + 6$ in completed square form i.e. $(x + g)^2 + h$.
(b) State the coordinates of the vertex (or turning point) of $y = x^2 - 2x + 6$.

Notes

When a quadratic is expressed in completed square form, the following information can be easily extracted:

- Coordinates of the ***vertex*** (or ***turning point*** or ***maximum/minimum***)
- Equation of the line of symmetry ($x =$ the $x-$coordinate of the vertex)
- Greatest (or least) value of the curve (the $y-$value of the vertex)

N.B. Whether it is the greatest or least value depends on whether the curve is concave-up or concave-down:

Concave-***up*** $\Rightarrow$ ***minimum*** so ***least*** value (+ve sign in front of bracket)
Concave-***down*** $\Rightarrow$ ***maximum*** so ***greatest*** value (—ve sign in front of bracket)

In general, for the equation $y = a(x + g)^2 + h$:

- The vertex is at $(-g,\ h)$
- Equation of line of symmetry is $x = -g$
- Greatest (or least) value is h

N.B. The greatest (or least) value is when the bracket = 0 — this is because a squared number is always positive
The value of a does not affect vertex, the line of symmetry or the greatest/least value

E.g. 1 Consider the graph of $y = (x - 2)^2 + 5$. State:
(a) the coordinates of the vertex
(b) whether the vertex is a maximum or minimum
(c) the equation of the line of symmetry
(d) the greatest or least value, indicating whether it is the greatest or least value

Working:
(a) Vertex is at $(2, 5)$
(b) +ve sign in front of bracket so concave-up ∴ a minimum
(c) Line of symmetry: $x = 2$ ***vertical line through the vertex***
(d) Minimum so 5 is the least value

E.g. 2 For these quadratic curves state (i) the coordinates of the vertex, (ii) the equation of the line of symmetry and (iii) the greatest or least value, indicating whether it is a greatest or least value.

(a) $y = 5(x+4)^2 - 8$ (b) $y = 7 - 3(x-1)^2$

(c) $y = -1 - (x+8)^2$ (d) $y = 6(x-9)^2 + 2$

Success criteria — completing the square

The method for completing the square is easier when the coefficient of x^2 is 1:

Coefficient of x^2 is 1:

1. Halve the coefficient of x to get the number in the bracket.
2. Subtract the square of the number in the bracket.
3. Include the constant term at the end

E.g. 3 Express $x^2 - 2x + 10$ in completed square form:

Working: Half of −2 is −1: so $x^2 - 2x + 10 \equiv (x-1)^2 \dots$

Subtract the square of the number in the bracket: $(x-1)^2 - (-1)^2 \dots$

Include the constant term at the end: $(x-1)^2 - (-1)^2 + 10$

$$\begin{aligned} x^2 - 2x + 10 &\equiv (x-1)^2 - (-1)^2 + 10 \\ &= (x-1)^2 + 9 \end{aligned}$$

N.B. $-(-1)^2 = -1$: we always take something away from the bracket because the bracket has added an extra term

E.g. 4 Express these quadratic expressions in completed square form:

(a) $x^2 + 4x + 20$ (b) $x^2 - 9x - 10$

Coefficient of x^2 is not 1, including when it is negative:

1. Factorise the coefficient of x^2 out of the terms in x^2 and x
2. Complete the square of the expression in the bracket (i.e. halve the coefficient of x to get the number in the bracket and subtract the square of the number in the bracket, include the constant term at the end)
3. Multiply the terms in the bracket by the number outside the bracket
4. Simplify

E.g. 5 Express these quadratic expressions in completed square form:

(a) $2x^2 - 16x + 9$ (b) $1 - 2x - x^2$

Working: (a)

$$\begin{aligned} 2x^2 - 16x + 9 &\equiv 2\left[x^2 - 8x\right] + 9 && \textit{factorise the coefficient of } x^2 \\ &= 2\left[(x-4)^2 - (-4)^2\right] + 9 && \textit{complete the square} \\ &= 2\left[(x-4)^2 - 16\right] + 9 \\ &= 2(x-4)^2 - 32 + 9 && \textit{expand square brackets} \\ &= 2(x-4)^2 - 23 \end{aligned}$$

(b)

$$\begin{aligned} 1 - 2x - x^2 &\equiv 1 - \left[x^2 + 2x\right] && \textit{factorise the coefficient of } x^2 \\ &= 1 - \left[(x+1)^2 - 1^2\right] && \textit{complete the square} \\ &= 1 - \left[(x+1)^2 - 1\right] \\ &= 1 - (x+1)^2 + 1 && \textit{expand square brackets} \\ &= 2 - (x+1)^2 \end{aligned}$$

E.g. 6 For these quadratic expressions (i) express them in completed square form (ii) state the coordinates of the turning point:
(a) $y = 2x^2 + 12x + 5$ (b) $y = 5 + 10x - x^2$

E.g. 7 Find the equation of the quadratic in the form $y = ax^2 + bx + c$ given that the vertex is at $(-3, 12)$ and the curve passes through the point $(1, -4)$.

E.g. 8 Given that $g(x) = x^2 + 8x + 20$, show that $g(x) \geq 4$ for all values of x.

From completed square form the quadratic equation can also be solved.

E.g. 9 Solve the equation $x^2 - 7x - 1 = 0$ by completing the square. Give your answers exactly.

Video: Completing the square
Video: Applications of completing the square

Solutions to Starter and E.g.s

Exercise
p40 3C Qu 1-3 (i), 4-9, (10)

Summary
Completing the square when the coefficient of x^2 is not 1, including when it is negative:
1. Factorise the coefficient of x^2 out of the terms in x^2 and x
2. Complete the square of the expression in the bracket (i.e. halve the coefficient of x to get the number in the bracket and subtract the square of the number in the bracket, include the constant term at the end)
3. Multiply the terms in the bracket by the number outside the bracket
4. Simplify