Converting Exponentials to a Linear Model

Starter

1. (Review of last lesson) A fungus is being grown under controlled conditions in a laboratory. Initially, it covers an area of 4 mm². After t hours, its are is N mm², where $N = N_0 e^{kt}$ and N_0 and k are constants. After 6 hours its area is 10 mm².
   (a) Find the values of N_0 and k.
   (b) Predict the area of the fungus after 12 hours.
   (c) How long will it take for the fungus to grow to 15 mm²?
   (d) Describe one limitation of the model used.

Notes

In the starter question, two measurements were taken: one at the start when $t = 0$ and another when $t = 6$. In reality it is likely that a scientist would take more measurements so that the model becomes more accurate.

Let us imagine that a scientist took the following measurements (include a 3rd row for later).

t	0	1	2	3	4	5	6
N (measured)	4	5.2	5.9	6.9	7.8	9.1	10

To take these extra measurements into account, the equation $N = N_0 e^{kt}$ is turned into a ***linear relationship*** so that the data points can be plotted and a ***line of best fit*** drawn. The gradient and $y-$intercept of the line can then be found and these can be used to find N_0 and k.

The first step is to take logs of both sides, but since e is involved in the equation take $\ln$ of both sides.

Initial relationship: $N = N_0 e^{kt}$
Take $\ln$ of both sides: $\ln N = \ln\left(N_0 e^{kt}\right)$
1st law of logs: $\ln N = \ln N_0 + \ln e^{kt}$
3rd law of logs: $\ln N = kt \ln e + \ln N_0$ ***order of terms on RHS also swapped***
Since $\ln e = 1$: $\ln N = kt + \ln N_0$

The variables are t and N. By letting $y = \ln N$ and $x = t$, we get $y = kx + \ln N_0$ which is the equation of a straight line where:

Horizontal axis: t
Vertical axis: $\ln N$
Gradient: k
$y-$intercept: $\ln N_0$

In order to plot this graph we need the values of $\ln N$ (this is what the 3rd line of the table is for).

t	0	1	2	3	4	5	6
N (measured)	4	5.2	5.9	6.9	7.8	9.1	10
$\ln N$	1.4	1.65	1.8	1.9	2.05	2.2	2.3

N.B. Given the graph will be plotted on graph paper, 1 d.p. accuracy is fine.

E.g. 1 (a) On graph paper, draw the graph of $\ln N$ ($y-$axis) against t ($x-$axis) and plot the points from the table.

(b) Draw a line of best fit through the points.
N.B. There should be roughly an equal number of points above and below the line.

(c) Calculate the gradient of the straight line and the note the $y-$intercept.
N.B. Each person's graph will be slightly different so the values of the gradient and $y-$intercept will be slightly different.

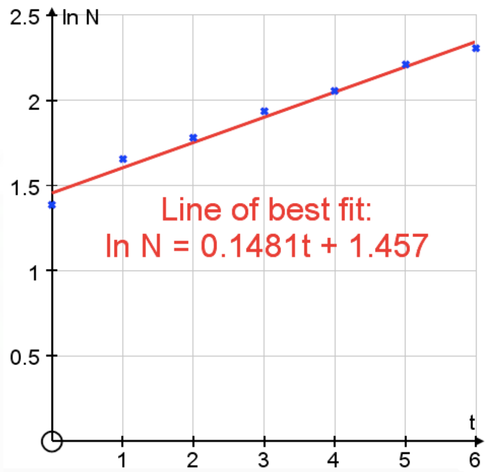
(d) Using your answers to (b), find the values of N_0 and k. Hence write down the exponential formula involving N and t.

Working: (a) and (b) See graph

(c) Gradient $\approx 0.15\ (\pm 0.02)$
$y-$intercept $\approx 1.46\ \pm 0.15$

(d) Gradient $= k$
So $k \approx 0.15\ (\pm 0.02)$

$y-$intercept = $\ln N_0$
so $N_0 = e^{\text{y-intercept}}$
$N_0 \approx e^{1.46}$
$N_0 \approx 4.3$
($3.7 \leq N_0 \leq 5.0$ is ok)



Formula is $N = 4.3e^{0.15t}$ (compared to $N = 4e^{0.153t}$)

Finding the equation of the line of best fit on your calculator
Instead of drawing the graph, the equation of the line of best fit can be got from your calculator.

Menu >> 6: Statistics = >> 2: y = a + bx >> Enter data >> OPTN >> 4

Video: **Equation of line of best fit using a Classwiz calculator**

E.g. 2 Use your calculator to find the equation of the line of best fit for the data above.
N.B. Enter the $t-$values in the first column and $\ln N$ in the second column.

Working: $y = a + bx$ where $a = 1.45656$ and b = 0.1481

2-values: $N = 4e^{0.153t}$
7-values: $N = 4.29e^{0.148t}$ – a better model
%age difference = 3.20%

Additional cases

Express the function as a linear relationship using $\log_{10}$.

N.B. $\ln$ could also be used but since e is not included in the following, we tend to use $\log_{10}$

1. Let $N = N_0 a^t$.

e is not included in the formula so use $\log_{10}$

Initial relationship: $N = N_0 a^t$

Take $\log$**s of both sides**: $\log N = \log(N_0 a^t)$

1st law of logs: $\log N = \log N_0 + \log(a^t)$

3rd law of logs: $\log N = t \log a + logN_0$

Horizontal axis: t

Vertical axis: $\log N$

Gradient: $\log a$

$y-$intercept: $\log N_0$

2. Let $N = At^k$.

e is not included in the formula so use $\log_{10}$

Initial relationship: $N = At^k$

Take $\log$**s of both sides**: $\log N = \log(At^k)$

1st law of logs: $\log N = \log A + \log t^k$

3rd law of logs: $\log N = k \log t + \log A$

Horizontal axis: $\log t$

Vertical axis: $\log N$

Gradient: k

$y-$intercept: $\log A$

N.B. The axes involve your variable N and t

The gradient and $y-$intercept involve the constants.

Summary

Original	Linear form	Horizontal axis	Vertical axis	Gradient	y-intercept
$N = N_0 e^{kt}$	$\ln N = kt + \ln N_0$	t	ln N	k	$\ln N_0$
$N = N_0 a^t$	$\log N = t \log a + logN_0$	t	log N	$\log a$	$\log N_0$
$N = kt^n$	$\log N = n \log t + \log k$	log t	log N	k	$\log A$

E.g. 3 The number of bacteria, p, in a petri dish is observed over a period of time, t. The bacteria population can be modelled over time by the formula $p = at^b$, where a and b are constants. The results from the observations are shown in the table below.

t (days)	1	3	4	6	9
p (1000s)	2	14	22	44	88

Plot a linear graph to represent this data and use this to find the values of a and b.

E.g. 4 Two variable, S and x, satisfy the formula $S = 4 \times 7^x$.

(a) Find a linear relationship connecting S and x.

(b) The straight line graph of $\log S$ against x is plotted. Write down the gradient and the value of the intercept on the vertical axis.

Video: Modelling exponential curves - converting to linear form (3 videos)

Solutions to Starter and E.g.s

Exercise

p143 8D Qu 1, 2, 5, 6, 7 (no graph paper required)

Reducing exponentials to linear form 1 (Modelling) Qu 3-5 (need 1 sheet of A5 graph paper)

Summary

Original	Linear form	Horizontal axis	Vertical axis	Gradient	y-intercept
$N = N_0 e^{kt}$	$\ln N = kt + \ln N_0$	t	ln N	k	ln N_0
$N = N_0 a^t$	$\log N = t \log a + log N_0$	t	log N	$\log a$	$\log N_0$
$N = kt^n$	$\log N = n \log t + \log k$	log t	log N	k	$\log A$