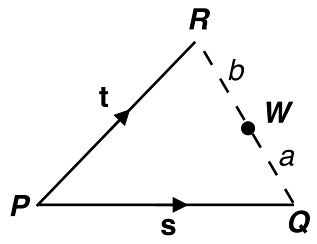
Differentiating from First Principles

Starter

1. (Review of last lesson) $\overrightarrow{PQ} = \mathbf{s}$ and $\overrightarrow{PR} = \mathbf{t}$. The point W lies on QR and divides it in the ratio $a : b$. Given that $\overrightarrow{PW} = \frac{5}{9}\mathbf{s} + \frac{4}{9}\mathbf{t}$, find the values a and b.



Notes

Differentiation forms part of calculus along with integration.

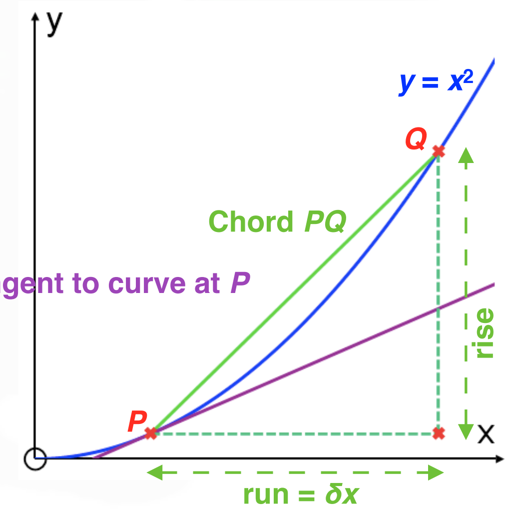
Differentiation is all about gradients of curves and seeing what information we can get from the gradient of a curve. Straight lines have constant gradients but curves have gradients that change. At GCSE we estimated the gradient of a curve by drawing a tangent at the curve, forming a right-angled triangle and using the formula $\text{Gradient} = \dfrac{\text{change in y}}{\text{change in x}}$. We need a method that is quick and accurate.

The following ideas were developed independently by Sir Isaac Newton and Gottfried Leibniz towards the end of the 17th century.

Key language

The ***gradient function*** is called the (first) ***derivative***.
The ***process*** to get the (first) derivative is called ***differentiation***.

See Differentiation from first principles notes



Gradient of the curve $y = x^2$

We aim to find the gradient of the curve $y = x^2$ at the point $P(x, y) \equiv (x, x^2)$.

The purple tangent has the same gradient as the curve at P.

Let Q also be on the curve $y = x^2$.

The gradient of the chord PQ approximates the gradient of the curve at P.

As Q gets closer and closer to P the gradient of the chord PQ gets closer to the gradient of the tangent at P.

Let the $x-$coordinate of Q be $x + \delta x$ where δx is a small amount of x i.e. δx is close to zero.

Since Q lies on $y = x^2$, its $y-$coordinate is $(x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2$.

So $P(x,\ x^2)$ and $Q(x + \delta x,\ (x + \delta x)^2)$.

$$\text{Gradient of chord PQ} = \frac{\text{rise}}{\text{run}} = \frac{(x+\delta x)^2 - x^2}{x+\delta x - x}$$

$$= \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$ ***expand the brackets***

$$= \frac{2x\delta x + (\delta x)^2}{\delta x}$$ ***simplify***

$$= \frac{\delta x(2x + \delta x)}{\delta x}$$ ***factorise***

$$= 2x + \delta x$$

As $\delta x \to 0$, the gradient of the chord $PQ \to$ gradient of the tangent at $P =$ gradient of the curve at P.
As $\delta x \to 0$, the gradient of the chord $PQ \to 2x$

Therefore the gradient of the curve $y = x^2$ is $2x$.

Show on spreadsheet 'Investigating gradient of y = x^2'.

Differentiation from 1st principles

To differentiate from 1st principles we use the formula: $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{x + \delta x - x}$

N.B. $f(x+\delta x)$ means replace x by $x + \delta x$ in the function.

What does $\lim_{\delta x \to 0}$ ***mean?***

$\lim_{\delta x \to 0}$ means **'the limit as h tends to zero"**

i.e. we can't evaluate the expression when $\delta x = 0$, but as δx gets smaller the expression gets closer to a fixed (or limiting) value.

You may be asked to differentiate from first principles in the exam. Here are important pointers:

- The ***original function always disappears*** in the numerator (like x^2 above).
- You ***always need to take out a factor of δx out of the numerator*** before cancelling it with the δx in the denominator
- $\lim_{\delta x \to 0}$ needs to ***remain until your final answer***.

E.g. 1 Differentiate the function $f(x) = x^2 - 7x$ from first principles.

Working:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{x + \delta x - x}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{(x + \delta x)^2 + 7(x + \delta x) - (x^2 + 7x)}{x + \delta x - x}$$ ***replace x by $x + \delta x$***

$$f'(x) = \lim_{\delta x \to 0} \frac{x^2 + 2x\delta x + (\delta x)^2 + 7x + 7\delta x - x^2 - 7x}{\delta x}$$ ***expand***

$$f'(x) = \lim_{\delta x \to 0} \frac{2x\delta x + (\delta x)^2 + 7\delta x}{\delta x}$$ ***original function $x^2 - 7x$ disappears***

$$f'(x) = \lim_{\delta x \to 0} \frac{\delta x(2x + \delta x + 7)}{\delta x}$$ ***factorise δx out***

$$f'(x) = \lim_{\delta x \to 0} (2x + \delta x + 7)$$ ***cancel δx in the numerator and denominator***

$$f'(x) = 2x - 7$$ ***remove $\lim_{\delta x \to 0}$ and δx***

Questions may require you to use the binomial theorem

E.g. 2 Differentiate the function $f(x) = 4x^3$ from first principles.

Video: Differentiation from 1st principles (2nd video)

Solutions to Starter and E.g.s

Exercise
p253 13B Qu 3, 5, 8, 9

Summary
The ***gradient function*** is called the (first) ***derivative***.
The ***process*** to get the (first) derivative is called ***differentiation***.
To differentiate from 1st principles we use the formula: $f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{x + \delta x - x}$