Exponential Modelling

Starter

1. **(Review of a previous lesson)**

Solve, giving your answers exactly: (a) $e^{x-3} = 14$ (b) $5e^{3x} = 45$.

N.B. For (b), "Take $\ln$ of both sides" since the equation includes e.

Notes

Exponential growth/decay is when the rate of growth/decay depends on how much of the substance there is.

- More substance $\Rightarrow$ faster growth/decay
- Less substance $\Rightarrow$ slower growth/decay

N.B. "Find the rate of growth/decay" means find the gradient at that point.
If $N = Ae^{kt}$ then the gradient of N is ke^{kt} (or kN)
$k > 0 \Rightarrow$ growth
$k < 0 \Rightarrow$ decay

E.g. 1 The concentration (C) of a drug in the bloodstream, t hours after taking an initial dose, decreases exponentially according to $C = Ae^{-kt}$, where A and k are constants.

(a) If the initial concentration is 0.72, and this halves after 5 hours, find the values of A and k.

(b) Find the rate of change of the concentration when $t = 2$

(c) Sketch the graph of C against t.

Working:

(a) Initial concentration is 0.72 so when $t = 0$, $C = 0.72$

$72 = Ae^{-k\times 0} \Rightarrow A = 0.72$ since $e^0 = 1$

When $t = 5$, $C = \dfrac{0.72}{2} = 0.36$

$0.36 = 0.72e^{-5k} \Rightarrow 0.5 = e^{-5k}$

$\ln 0.5 = \ln e^{-5k} \Rightarrow \ln 0.5 = -5k$ *since* $\ln e = 1$

$-\dfrac{1}{5}\ln 0.5 = k \Rightarrow k = 0.139$ (3 s.f.)

$C = 0.72e^{-0.139t}$

(b) Rate of change of concentration is given by $-0.139 \times 0.72e^{-0.139t}$

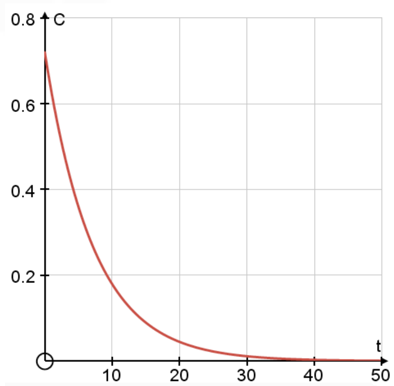
When $t = 2$, rate of change is $-0.139 \times 0.72e^{-0.139\times 2}$

So rate of change is -0.0758.

(c) $t \geq 0$

When $t = 0$, $C = 0.72$

Exponentially decay



E.g. 2 The exponential growth of a colony of bacteria can be modelled by the equation $B = 60e^{0.03t}$, where B is the number of bacteria and t is the time in hours from the point the colony is first monitored ($t \geq 0$). Use the model to:

(a) Work out the initial population of bacteria.

(b) Predict the number of bacteria after 4 hours.

(c) The growth of a different bacteria is modelled by the function $25e^{0.1t}$. Compare the two population models.

E.g. 3 £350 is initially paid into a bank account that pays 3% interest per year. No further money is deposited or withdrawn from the account. Write down an equation to show how much money will be in the account after t years. Use your model to calculate how many whole years it will take before there is more than £1000 in the account.

Working: Adding 3% each year $\Rightarrow \quad \times 1.03$

Money in the account after t years $= 350 \times 1.03^t$

$350 \times 1.03^t > 1000$

$1.03^t > \dfrac{1000}{350}$

Take logs of both sides: $\log 1.03^t > \log \dfrac{100}{35}$

$t \log 1.03 > \log \dfrac{20}{7}$

$t > \dfrac{\log 20 - \log 7}{\log 1.03}$

N.B. Since $\log 1.03 > 0$, the inequality sign does not change direction

$t > 35.5$

It will take 36 years to reach £1000 in the account.

Limitations of modelling:

- Models simplify a situation by ignoring some factors
- Exponential models only match the real-world for a short amount of time. For example, there may be an upper limit, while the model increases towards infinity

E.g. 4 The penguin population, P, of a small island can be modelled by the formula $P = 5000e^{0.1t}$, where t is the number of years after the initial survey.

(a) What does 5000 represent in the formula.

(b) Explain why this model may not be appropriate for the long term.

Working: (a) 5000 is the initial number of penguins i.e. when $t = 0$.

(b) The model suggests the population will tend to infinity (e.g. after 60 years the populations will be over 2 million). This is unrealistic as it does not take into other factors such as finite food supply, predators or possible disease from overcrowding.

N.B. Rate of growth/decay means find the gradient at that point

Video: Exponential growth problem
Video: Exponential growth and decay problems

Solutions to Starter and E.g.s

Exercise
p139 8C Qu 2-7, 9 (10-12)

Summary

Exponential growth/decay — rate of growth/decay depends on amount of substance.

- More substance ⇒ faster growth/decay
- Less substance ⇒ slower growth/decay

"Find the rate of growth/decay" means find the gradient at that point.

If $N = Ae^{kt}$ then the gradient of N is ke^{kt} (or kN)

$k > 0$ ⇒ growth

$k < 0$ ⇒ decay

Limitations of modelling:

Whats factors have been ignored?

Does the model only fit for a limited amount of time?

What happens when time gets very big?