Geometrical Significance of Definite Integration

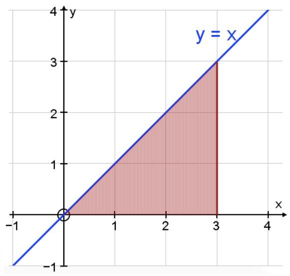
Starter

1. **(Review of last lesson)** Let $\int_1^4 f(x)dx = 8$. Find $\int_1^4 (5f(x) + 2x + 3)dx$.

2. **(Review of last lesson)**

Find the two possible values for A that satisfy $\int_{-2}^{2} \left(\frac{21}{8}x^2 + \frac{A}{x^2}\right)dx = 3A^2$.

Notes

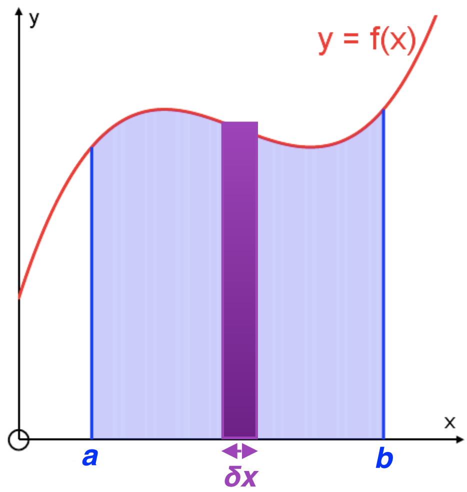


We know that $\int_0^3 x\,dx = \left[\frac{1}{2}x^2\right]_0^3 = \frac{1}{2}\left(3^2 - 0^2\right) = 4.5$,
but what does this number mean?

By considering the graph of $y = x$ from 0 to 3, we can see that the area between the line and $x-$axis is equal to $\frac{1}{2} \times 3 \times 3 = 4.5$.

The meaning of definite integration
Reminder from differentiation:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$



Imagine we are trying to find the area between the curve $y = f(x)$, the $x-$axis and the vertical lines $x = a$ and $x = b$.

If the total area is A, a small amount of that area, δA, is given by the purple rectangle. The width of this rectangle is δx and is very small.

Since $y = f(x)$ is a curve and δA is a rectangle, the latter is an approximation of the area of between the curve and the $x-$axis.

i.e. $\delta A \approx y \times \delta x$

where y is the height of the rectangle.

$$\frac{\delta A}{\delta x} \approx y$$

As the width of the rectangle gets smaller and smaller (i.e. as $\delta x \to 0$), the area δA gets closer and closer to the required area.

$$\therefore \lim_{\delta x \to 0} \frac{\delta A}{\delta x} = y$$

But $$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} = \frac{dA}{dx}$$

So $$\frac{dA}{dx} = y$$

Integrating both sides and include the limits from a to b: $A = \int_a^b y\,dx$

Definite integration finds the area between the curve and the $x-$axis and the $x-$values given as limits on the integration symbol.

E.g. 1 Find the area between the curve $y = x^2 + 3$, the x-axis and the lines $x = 1$ and $x = 2$.

Working:

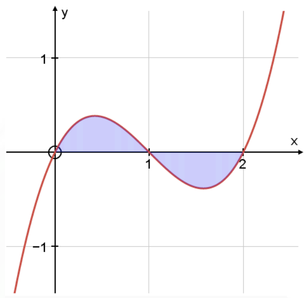
$$\int_1^2 (x^2+3)dx = \left[\frac{1}{3}x^3 + 3x\right]_1^2$$
$$= \left(\frac{1}{3}\times 2^3 + 3\times 2\right) - \left(\frac{1}{3}\times 1^3 + 3\times 1\right)$$
$$= \frac{16}{3} = 5\frac{1}{3}$$

E.g. 2 Find the area between the curve $y = \sqrt{x}$, the $x-$axis and the lines $x = 4$ and $x = 9$.

E.g. 3 Find the area between the curve $y = x^3 - 3x^2 + 2x$, the $x-$axis for $x = 0$ and $x = 2$.
Hence, or otherwise, sketch the graph of $y = x^3 - 3x^2 + 2x$.

Working:

$$\int_0^2 x^3 - 3x^2 + 2x = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^2$$
$$= \left(\frac{1}{4}\times 2^4 - 2^3 + 2^2\right) - \left(0\right) = 0$$



The answer of zero causes some students to believe the curve must run along the $x-$axis between 0 and 2.

In fact, as can be seen from the diagram, the areas above and below the $x-$axis are equal and cancel each other out since ***integrations*** where the function is ***below the $x-$axis*** come out as ***negative***.

Therefore to get the answer, we do separate integrations:

$$\int_0^1 (x^3 - 3x^2 + 2x)dx = \frac{1}{4} \quad \text{and} \quad \int_1^2 (x^3 - 3x^2 + 2x)dx = -\frac{1}{4}$$

Since area only takes positive values, the negative value is made positive before adding:

Required area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

How to avoid the "negative area" trap

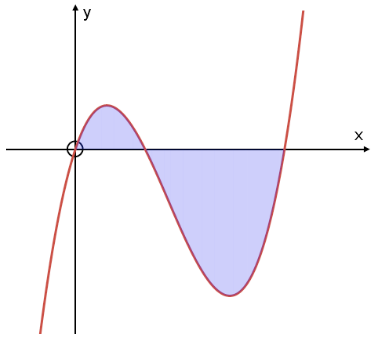
If a sketch of the curve is not given in the question, the easiest way to see if the definite integration needs be to split into 2 or more integrations is to ***find the roots of the curve***.

If one or more roots lie between the limits of the integration, the area will need to be split into separate integrations — any that give negative values should be made positive before adding on the values together.

Let there be a root at $x = p$ for the graph of the curve $y = f(x)$ such that $a < p < b$, then

$$\int_a^b f(x)dx = \left|\int_a^p f(x)dx\right| + \left|\int_p^b f(x)dx\right|$$

E.g. 4 Calculate the exact value of the shaded area shown for the curve $y = x^3 - 4x^2 + 3x$.



E.g. 5 Find the area enclosed by the curve with equation $y = 3x^2 + 6x - 9$, the $x-$axis and the lines $x = -2$ and $x = 2$.

Video: Area bound by curve and x-axis

Area bound by curve and x-axis EQ

Solutions to Starter and E.g.s

Exercise
p306 15E Qu 1i, 2i, 3-10

Summary
Definite integration finds the area between the curve and the $x-$axis and the $x-$values given as limits on the integration symbol.
Area below the $x-$axis comes out as a negative number following integration. Therefore, it is important to sketch the curve to see if it goes below the $x-$axis between the limits. To find the total area, make any negative numbers positive before adding.