Nature of Stationary Points

Starter

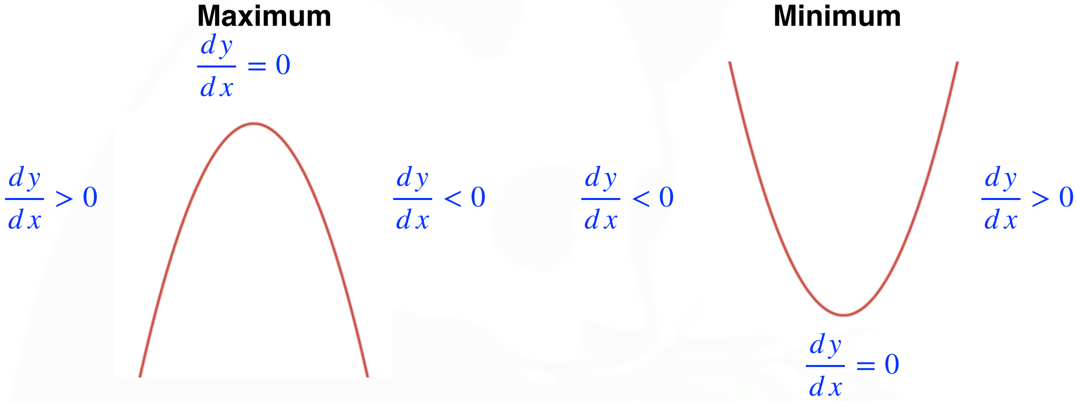
1. **(Review of last lesson)** The curve given by $f(x) = x^3 + ax^2 + bx + c$ has a stationary point at $(3,\ 10)$. Given also that $f''(3) = 0$, find a, b and c.

Notes

There are essentially two types of stationary point — maximums and minimums.

What can we say about the gradient of the curve either side of the stationary point?

Maximum **Minimum**



Gradient change: **+ve / 0 / —ve** **—ve / 0 / +ve**

N.B. The zero refers to the gradient at the stationary point itself.

Second derivative: $\frac{d^2y}{dx^2} < 0$ (gradient is decreasing) $\frac{d^2y}{dx^2} > 0$ (gradient is increasing)

There are two methods to determine the nature of a stationary point (i.e. decide whether it is a maximum or a minimum) — the ***second derivative*** or the ***gradient change*** methods.

The second derivative method is usually the quickest but the gradient change method is useful to fall back on when a function is particularly hard to differentiate.

Second derivative method

Let's assumed the $x-$value of a stationary point has been found.
Substitute the $x-$value of the stationary point into the expression for the second derivative.

$$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \quad \text{maximum}$$
$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad \text{minimum}$$

N.B. If $\frac{d^2y}{dx^2} = 0$ at the stationary point then there is no information on the stationary point. Use the gradient change method to check whether it is maximum or a minimum.

The situation $\frac{d^2y}{dx^2} = 0$ is covered in depth in the A2 maths course.

Gradient change method

This method is slightly more complicated than the second derivative method. Let's assume the stationary point has been found.

1. Choose $x-$values just to the left and to the right of the stationary point (e.g. if $x = 2$ is the stationary point, then $x = 1.9$ and $x = 2.1$ would be good).
2. Substitute these $x-$values into the first derivative, $\frac{dy}{dx}$, and note the signs (the actual values are not important):

   Gradient change: +ve / 0 / −ve ⇒ maximum

   −ve / 0 / +ve ⇒ minimum

N.B. Always substitute the point to the left of the SP first.

E.g. 1 The stationary points of the curve $y = x^3 - 15x^2 + 48x + 7$ are (2, 51) and (8, −57).

(a) Use the second derivative method to determine the nature of the stationary point $(2, -51)$.

(b) Use the gradient change method to determine the nature of the stationary point $(8, -57)$.

Working: (a)

$$\frac{dy}{dx} = 3x^2 - 30x + 48$$

$$\frac{d^2y}{dx^2} = 6x - 30$$

When $x = 2$, $\frac{d^2y}{dx^2} = 6 \times 2 - 30 < 0$

$\therefore (2, -51)$ is a maximum

(b)

$$\frac{dy}{dx} = 3x^2 - 30x + 48$$

When $x = 7.9$, $\frac{dy}{dx} = 3 \times 7.9^2 - 30 \times 7.9 + 48 < 0$

When $x = 8.1$, $\frac{dy}{dx} = 3 \times 8.1^2 - 30 \times 8.1 + 48 > 0$

Gradient change: −ve / 0 / +ve

$\therefore (8, -57)$ is a minimum

In general, use the second derivative method unless a function is particularly difficult to differentiate or $\frac{d^2y}{dx^2} = 0$ at the stationary point.

E.g. 2 Find and classify the stationary points of these curves:

(a) $y = 4x^3 + 3x^2 - 6x - 1$

(b) $y = 3x^4 - 8x^3 + 6x^2 - 3$

(c) $y = x^2 + \frac{16}{x^2}$

E.g. 3 Show that the graph of the function $f(x) = x^5 + 3x + 2$ has no stationary points.

Video: Nature of stationary points

Solutions to Starter and E.g.s

Exercise
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Summary
Use either the ***second derivative*** or ***gradient change*** methods to decide whether a stationary point is a maximum or a minimum.

Second derivative method
Substitute the $x-$value of the stationary point into the expression for the second derivative.

$$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \quad \text{maximum}$$

$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad \text{minimum}$$

Gradient change method

1. Choose $x-$values just to the left and to the right of the stationary point (e.g. if $x = 2$ is the stationary point, then $x = 1.9$ and $x = 2.1$ would be good).
2. Substitute these $x-$values into the first derivative, $\frac{dy}{dx}$, and note the signs (the actual values are not important):
   Gradient change: +ve / 0 / —ve ⇒ maximum
   —ve / 0 / +ve ⇒ minimum

N.B. Always substitute the point to the left of the SP first.