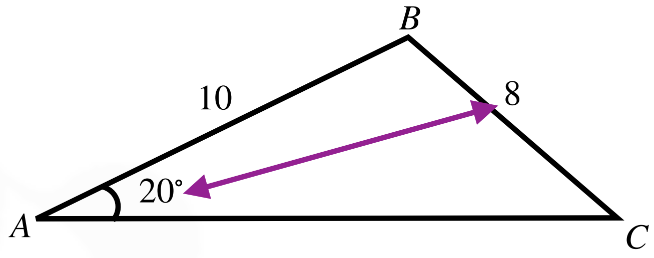
Sine rule, including the ambiguous case

Starter

1. *(Review of previous material)* A triangle ABC is such that $AB = 10$ cm, $BC = 8$ cm and $\angle BAC = 20°$. Using the sine rule, find the length of AC to 3 s.f..



2. (a) By considering the graph of $y = \sin x$, or otherwise, find two angles for $0° \leq \theta \leq 180°$ which satisfy $\sin\theta = 0.4275$
   (b) Hence find an alternative length for AB in the triangle from question 1.
   (c) Sketch the two triangles.

Notes

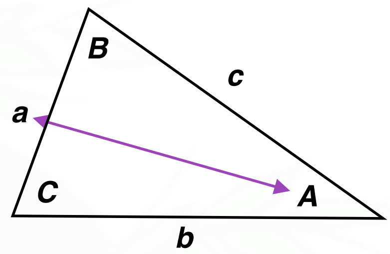
The sine rule was met in the GCSE course.

Notation reminder

A ***capital letter*** indicates an ***angle.***

A ***lower case*** letter indicates a ***side.***

A ***lower case side*** is opposite its ***capital lettered angle.***



Information necessary to use the sine rule

The sine rule is used when ***an angle and its opposite side*** are known.
If an arrow can be drawn between a known angle and its known opposite side, use the sine rule.

Finding angles: use $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

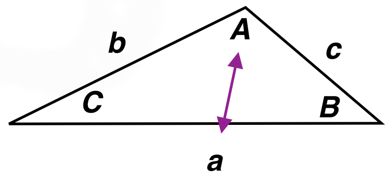
Finding sides: use $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Proof of sine rule

The area of the triangle is $\frac{1}{2}ab\sin C$.

We could also write the area as $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$.

So $\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$



Multiplying by 2 gives: $bc\sin A = ac\sin B = ab\sin C$

Dividing by abc gives: $\dfrac{bc\sin A}{abc} = \dfrac{ac\sin B}{abc} = \dfrac{ab\sin C}{abc}$

Cancelling letters: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

In practice, we just need two parts: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ — this is the ***sine rule***

We can flip the fractions to get: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Ambiguous case of the sine rule

The ambiguous case of the sine rule means that ***sometimes***, from the information given, two triangles can be drawn.

This is because when we solve an equation such as $\sin\theta = 0.5$ there are two possible angles for $0^\circ \le \theta \le 180^\circ$ since $\sin\theta \equiv \sin(180^\circ - \theta)$.

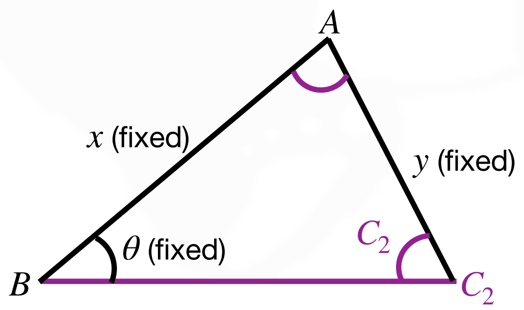
E.g. 1 In the triangle XYZ, $\angle X = 29.5^\circ$, $XY = 21$ cm and $YZ = 36$ cm. By calculation, decide whether the ambiguous case of the sine rule applies for this triangle.

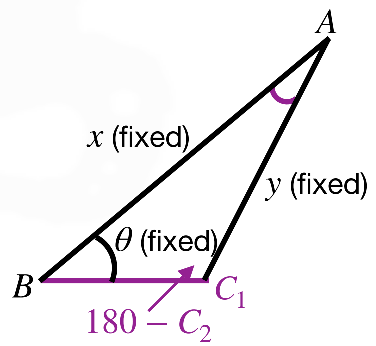
Geometric understanding of the sine rule

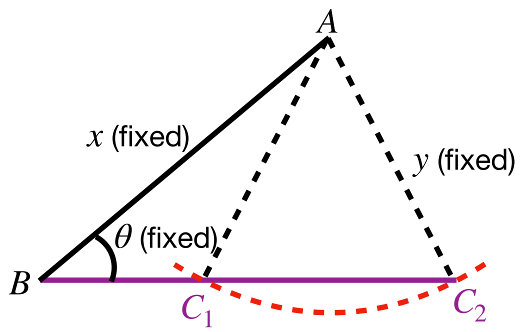
E.g. 2
(a) Draw a horizontal line about 10 cm long and label the left end A.
(b) From A, draw a side of length 8 cm at an angle of about 45° above the horizontal. Label the end of this line B.
(c) Open your compass to 6.5 cm. Put your compass point on B and draw an arc that cuts the horizontal line in two places. Label these points C_1 and C_2.
(d) Draw dotted lines from B to C_1 and B to C_2.
(e) Measure the lengths from A to the intersection points C_1 and C_2, giving your answers to the nearest millimetre.
(f) Measure the angles at $\angle ABC_1$ and $\angle ABC_2$.

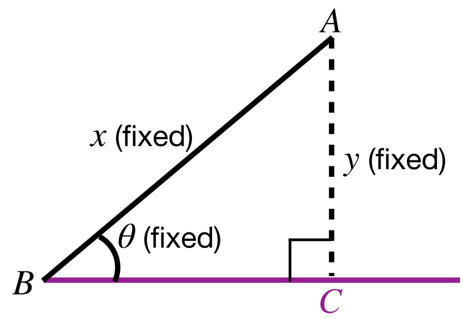
In the ambiguous case of the sine rule, ***two sides have fixed length*** and an ***angle is opposite one of the sides***. If, by putting the point of a pair of compasses on A, an arc can be drawn that cuts the horizontal side in two places, the ambiguous case of the sine rule exists for that triangle.

For the ambiguous case to exist, the length of the side opposite the given angle must be greater than the length needed to form a right angle.









Geogebra 1: Sine rule ambiguous case
Geogebra 2: Sine rule ambiguous case animation

E.g. 3 In the triangle XYZ, $YZ = 15$ and $\angle XYZ = 25^\circ$. Find the range of values of the length XZ, with $XZ < 15$ cm such that the ambiguous case exists for the triangle XYZ.

Video: Sine rule
Video: Sine rule - ambiguous case

Exam questions: Sine rule

Solutions to Starter and E.g.s

Exercise
p207 11A Qu 1i, 2i, 3, 4-5, (6-7 red)

Summary
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The ambiguous case of the sine rule means that ***sometimes***, from the information given, two triangles can be drawn. This is because $\sin\theta \equiv \sin(180^\circ - \theta)$.