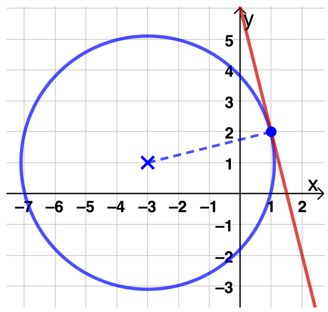
Solving problems with lines and circles

Starter

1. *(Review of previous material)*
Determine the nature of the roots of the curve with equation $y = 3x^2 - 5x + 7$.

2. Decide whether the line with equation $y = 7x + 10$ is a tangent to the circle $x^2 + y^2 = 2$.

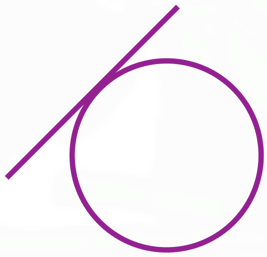
3. *(Review of GCSE material)* Find the equation of the tangent to the circle $(x + 3)^2 + (y - 1)^2 = 17$ at the point $(1, 2)$.
Hint: Find the gradient of the line segment joining the centre and $(1, 2)$ then…



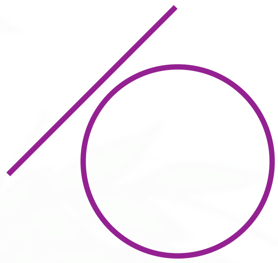
Notes

Since the powers of x and y in the equation of a circle are 2, when finding the ***intersection between a line and circle***, or ***two circles***, a quadratic equation is formed and the number of roots is given by the ***discriminant***.

2 points of intersection
$b^2 - 4ac > 0$



Line is tangent to circle
$b^2 - 4ac = 0$



No points of intersection
$b^2 - 4ac < 0$

E.g. 1 In how many places does the line $y = x + 1$ meet the circle given by the equation $x^2 + y^2 - 2x + 6y - 27 = 0$

Working: Replace y by $x + 1$:

$$x^2 + (x + 1)^2 - 2x + 6(x + 1) - 27 = 0$$
$$x^2 + x^2 + 2x + 1 - 2x + 6x + 6 - 27 = 0$$
$$2x^2 + 6x - 20 = 0$$
$$x^2 + 3x - 10 = 0$$

$a = 1 \qquad b = 3 \qquad c = -10$

$b^2 - 4ac = 3^2 - 4 \times 1 \times (-10) > 0$

So there are 2 points of intersection.

E.g. 2 The line with equation $x + y = k$ is a tangent to the circle $x^2 + y^2 + 4x - 6y + 11 = 0$.
Find the possible values of k.

Tangent to a circle at a point

Success criteria — finding the equation of a tangent to a circle

1. Find the ***gradient of the radius*** connecting the centre and the point on the circumference.
2. Find the gradient of the tangent by doing the ***negative reciprocal*** of the gradient of the radius.
3. Find the equation of the tangent using $y - y_1 = m(x - x_1)$ or using the $y = mx + c$ method.

E.g. 3 The equation of the circle C is $x^2 + y^2 - 2x + 6y = 27$. Find the equation of the tangent to C at the point $(2, 3)$, giving your answer in the form $ax + by = k$ where a, b and k are integers.

Working: $x^2 + y^2 - 2x + 6y = 27 \quad \Rightarrow \quad (x-1)^2 + (y+3)^2 = 37$

The centre is at $(1, \, -3)$

Gradient of radius $= \dfrac{-3-3}{1-2} = 6$

Gradient of the tangent is $-\dfrac{1}{6}$

Substitute $y - y_1 = m(x - x_1)$***:*** $\quad y - 3 = -\dfrac{1}{6}\left(x - 2\right)$

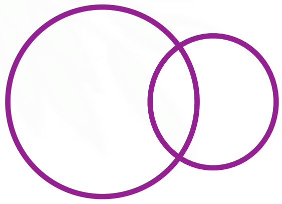
$6y - 18 = -x + 2$

The equation of the tangent is $x + 6y = 20$.

E.g. 4 Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 8y = 6$ at the point $(-1, 9)$, giving your answer in the form $ax + by = k$ where a, b and k are integers..

Intersecting circles

E.g. 5 In the diagram, the two circles intersect each other in two places. Without using the discriminant, state the conditions necessary for two circles:



(a) to intersect twice

(b) to intersect just once

(c) to not intersect at all

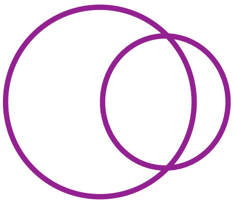
Assume the radii of the two circles are r_1 and r_2 where $r_1 \geq r_2$ and the distance between the centres of the circles is D.

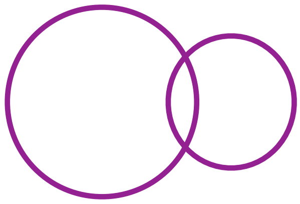
Working: (a) The distance between the centres is either:

less than the sum of the radii ***or***

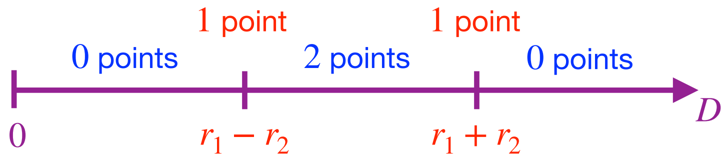
greater than the difference between the radii.

$r_1 - r_2 < D$ ***or*** $D < r_1 + r_2$





This diagram helps explain when two circles intersect, where D is the distance between the centre and r_1 and r_2 are the radii of the circles where $r_1 \geq r_2$



E.g. 6 Show that the circle $x^2 + y^2 - 10x - 6y + 30 = 0$ touches the circle $x^2 + y^2 + 6x - 6y - 18 = 0$.

E.g. 7* Find the range of values of k such that the circles given by $x^2 + y^2 - 8x - 12y = 12$ and $x^2 + y^2 + 4x - 28y = k$ intersect in two distinct places.

Working: $x^2 + y^2 - 8x - 12y = 12 \quad \Rightarrow \quad (x-4)^2 + (y-6)^2 = 64$

Centre is $(4, 6)$ and radius is 8

$x^2 + y^2 + 4x - 28y = k \quad \Rightarrow \quad (x+2)^2 + (y-14)^2 = k + 200$

Centre is $(-2, 14)$ and radius is $\sqrt{k+200}$

Distance between centres $= \sqrt{(4 - -2)^2 + (6-14)^2} = 10$

To intersect in two distinct places, either $r_1 - r_2 < D$ or $D < r_1 + r_2$

$r_1 - r_2 < D$:

$\sqrt{k+200} - 8 < 10 \quad \Rightarrow \quad k + 200 < 324 \quad \Rightarrow \quad k < 124$

$D < r_1 + r_2$:

$10 < \sqrt{k+200} + 8 \quad \Rightarrow \quad 4 < k + 200 \quad \Rightarrow \quad k > -196$

So $-196 < k < 124$

Video: **Equation of a tangent to a circle**

Exam questions: **Circles**

Solutions to Starter and E.g.s

Exercise

p107 6E Qu 1i, 2i, 3i, 4-6, 8-13, (14-15 red)

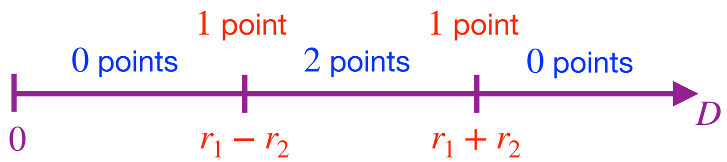
Summary

The number of ***points of intersection between a line and circle*** is found using the ***discriminant***.

Success criteria — finding the equation of a tangent to a circle:

1. Find the ***gradient of the radius*** connecting the centre and the point on the circumference.
2. Find the gradient of the tangent by doing the ***negative reciprocal*** of the gradient of the radius.
3. Find the equation of the tangent using $y - y_1 = m(x - x_1)$ or using the the $y = mx + c$ method.

The ***number of points of intersection between two circles*** is given by:



where D is the distance between the centre and r_1 and r_2 are the radii of the circles where $r_1 \geq r_2$.