Trigonometric identities

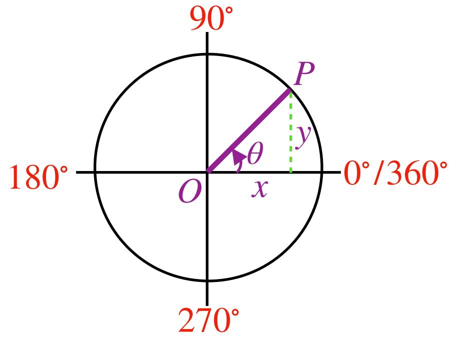
Starter

1. ***(Review of last lesson)*** Find the exact value of $\cos 30° + 2\sin 60°$.

Working: $\cos 30° + 2\sin 60° = \dfrac{\sqrt{3}}{2} + 2 \times \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{2}$

Notes

The unit circle was introduced in a previous lesson — it has a radius of 1 unit and is centred on the origin so its equation is $x^2 + y^2 = 1$. Points on its circumference have coordinates $P(\cos\theta, \sin\theta)$ where θ is the angle the line OP makes with the positive $x-$axis, measured in an anti clockwise direction.



Replacing x by $\cos\theta$ and y by $\sin\theta$ in $x^2 + y^2 = 1$ gives the trigonometric identity:

$$\sin^2\theta + \cos^2\theta \equiv 1$$

This identity is often more useful in rearranged form:

$$\cos^2\theta \equiv 1 - \sin^2\theta$$
$$\sin^2\theta \equiv 1 - \cos^2\theta$$

The second identity was also found in a previous lesson:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

An identity is true for all values of the unknown, which is θ in these cases. Identities should use the equivalent "≡" symbol rather than the equals "=" symbol.

E.g. 1 Simplify these expressions:

(a) $\dfrac{1-\sin^2\theta}{\cos^2\theta}$ (b) $\dfrac{\tan^2\theta}{1-\cos^2\theta}$ (c) $7\sin^2\theta + 7\cos^2\theta$

Working: (a) $\dfrac{1-\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta}{\cos^2\theta} = 1$

Proving identities

Additional identities can be proved using the two identities above or by using normal algebraic manipulation (e.g. adding algebraic fractions). When proving identities, ***start with the more complicated side***, usually the left-hand side, and ***simplify it step by step*** until the right-hand side is reached.

N.B. The right-hand side is only written in the final line of the working.

E.g. Prove the identity $\dfrac{1}{1+\cos\theta}+\dfrac{1}{1-\cos\theta}\equiv\dfrac{2}{\sin^2\theta}$

Working:

$$\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\equiv\frac{1-\cos\theta}{(1+\cos\theta)(1-\cos\theta)}+\frac{1+\cos\theta}{(1+\cos\theta)(1-\cos\theta)}$$

$$\equiv\frac{1-\cos\theta}{1-\cos^2\theta}+\frac{1+\cos\theta}{1-\cos^2\theta}$$

$$\equiv\frac{2}{1-\cos^2\theta}$$

$$\equiv\frac{2}{\sin^2\theta}$$

E.g. 2 Prove these identities:

(a) $\dfrac{\sin^2\theta}{1-\cos\theta}\equiv 1+\cos\theta$

(b) $\dfrac{1}{\tan\theta}+\tan\theta\equiv\dfrac{1}{\sin\theta\cos\theta}$

(c) $\dfrac{1-2\sin^2\theta}{\cos\theta+\sin\theta}\equiv\cos\theta-\sin\theta$

(d) $\sin\theta\tan\theta+\cos\theta\equiv\dfrac{1}{\cos\theta}$

Working: (a)

$$\frac{\sin^2\theta}{1-\cos\theta}\equiv\frac{1-\cos^2\theta}{1-\cos\theta}$$ ***since*** $\sin^2\theta\equiv 1-\cos^2\theta$

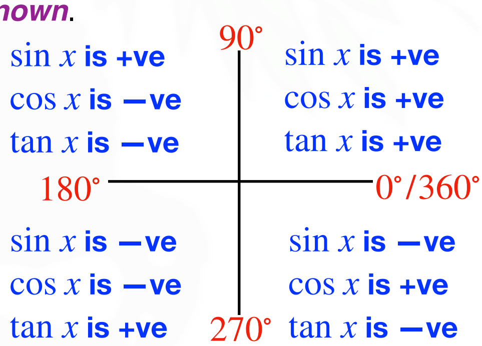
$$\equiv\frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}$$ ***difference of 2 squares***

$$\equiv 1+\cos\theta$$ ***cancelling***

The CAST diagram

Using ***exact values*** in trigonometry is useful so that accuracy is not lost during subsequent calculations. Exact trigonometry does not need to be restricted to angle such as 45° and 60° ***provided one of the trigonometric ratios of the angle is known***.

Methods can either employ Pythagoras' theorem or trigonometric identities, but it is useful to know what sign each trigonometric ratio is for angles between 0° and 360°

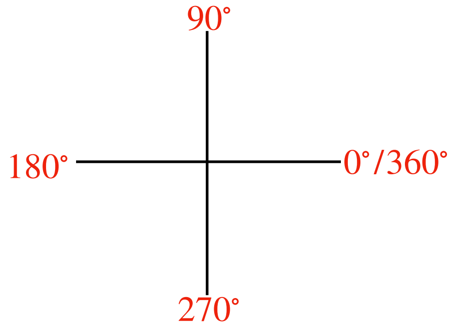


The diagram to the right gives the signs of the trigonometric ratios for each quadrant, however, it can be quite daunting to try and memorise.

An easier version is the CAST diagram, which simply indicates which trigonometric ratio is positive for each quadrant.

E.g. 3 The CAST diagram indicates which trigonometric ratio is positive for each quadrant.

(a) By using the diagram above, draw a similar diagram to the one on the right. Write in each quadrant the first letter of the trigonometric ratio(s) which is(are) positive in each quadrant.



(b) By using "A" for "All" in the quadrant where they are all positive, explain where the name "CAST diagram" comes from.

(c) Without a calculator, state the sign of the trigonometric ratio for the angle, or range of angles, given i.e. is is positive or negative?:

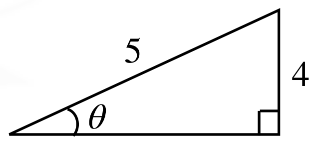
(i) $\tan\theta$ when $90^\circ < \theta < 180^\circ$ (ii) $\cos\theta$ when $270^\circ < \theta < 360^\circ$

(iii) $\sin 200^\circ$ (iv) $\cos 75^\circ$ (v) $\tan 320^\circ$ (vi) $\sin 160^\circ$

Exact value trigonometry

Here is an example with the working for both methods.

E.g. Given that $\sin\theta = \frac{4}{5}$, where $90^\circ < \theta < 180^\circ$, and without using a calculator, find the exact value $\cos\theta$ and $\tan\theta$.



Working: ***Pythagoras method:***

Ignoring the fact that $90^\circ < \theta < 180^\circ$ and since $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ a right-angled triangle can be drawn (opp $= 4$ and hyp $= 5$)

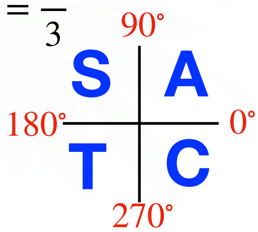
By Pythagoras the missing side is $\sqrt{5^2 - 4^2} = 3$.

So if θ was acute, $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$ and $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$

Now use the CAST diagram to decide the sign:

Only sine is positive for $90^\circ < \theta < 180^\circ$ so

$\cos\theta = -\frac{3}{5}$ and $\tan\theta = -\frac{4}{3}$



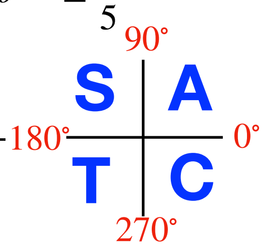
Identities method:

Since $\sin\theta = \frac{4}{5}$, use $\cos^2\theta \equiv 1 - \sin^2\theta$:

$$\cos^2\theta \equiv 1 - \left(\frac{4}{5}\right)^2$$
$$\cos^2\theta = \frac{9}{25}$$
$$\cos\theta = \pm\frac{3}{5}$$

The CAST diagram is then used to decide between the positive or negative value.

Only sine is positive for $90^\circ < \theta < 180^\circ$ so $\cos\theta = -\frac{3}{5}$



Using $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$: $\tan\theta = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$

N.B. θ is acute $\Rightarrow$ $0^\circ < \theta < 90^\circ$
θ is obtuse $\Rightarrow$ $90^\circ < \theta < 180^\circ$
θ is reflex $\Rightarrow$ $180^\circ < \theta < 360^\circ$

E.g. 4 Find the exact value of the other two trigonometric ratios given that:

(a) $\cos\theta = \dfrac{12}{13}$ and θ is acute (b) $\tan\theta = \dfrac{7}{24}$ and θ is reflex

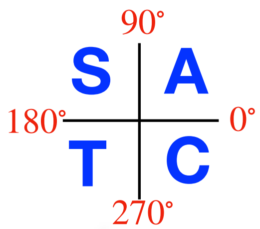
Video: Trigonometric identities (AS)
Video: Proving identities

Solutions to Starter and E.g.s

Exercise
p183 10D Qu 1i, 2, 3, 4i, 5-11, (12-13 red)

Summary
Trigonometric identities:

$$\sin^2\theta + \cos^2\theta \equiv 1$$
$$\cos^2\theta \equiv 1 - \sin^2\theta$$
$$\sin^2\theta \equiv 1 - \cos^2\theta$$
$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$



CAST diagram — the letters in tell us which trigonometric ratio is positive for the range of angles in that quadrant. A = All.

When ***proving identities, start with the more complicated side***, usually the left-hand side, and ***simplify it step by step*** until the right-hand side is reached.