Vector Geometry

Starter

1. **(Review of last lesson)**

The position vectors of 3 vertices of a parallelogram are $\begin{pmatrix}4\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}8\\6\end{pmatrix}$. Find the two possible position vectors of the 4th vertex in the first quadrant.

Notes

Collinear

How could we prove using vectors that three points, A, B and C are collinear?

Show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel i.e. are multiples of each other. Then state:

"Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel and share the common point A, the points A, B and C must be collinear."

E.g. 1 A, B and C are the points (2, 5), (4, 9) and (−3, −5).

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(b) Show that all three points are collinear.

Working: (a) $\overrightarrow{AB} = \begin{pmatrix}4\\9\end{pmatrix} - \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}-3\\-5\end{pmatrix} - \begin{pmatrix}4\\9\end{pmatrix} = \begin{pmatrix}-7\\-14\end{pmatrix}$

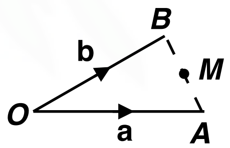
(b) $\overrightarrow{BC} = -\frac{7}{2}\overrightarrow{AB}$ therefore the vectors are multiples are one another.

This means that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

Since both vectors pass through B, the points are collinear.

Midpoints

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ and let M be the midpoint of AB. Express the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$.



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

Position vector of M $= \overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

E.g. 2 M is the midpoint of the line PQ, where P has position vector $-3\mathbf{i} + \mathbf{j}$ and M has position vector $2\mathbf{i} - 5\mathbf{j}$. What is the position vector of Q?

Hint: if you are not sure what to do, draw a diagram.

Working: Since $\overrightarrow{PM} = \overrightarrow{MQ}$, we can find the position vector of Q by adding $\overrightarrow{MQ}$ to the position vector of M

$\overrightarrow{PM} = \begin{pmatrix}2\\-5\end{pmatrix} - \begin{pmatrix}-3\\1\end{pmatrix} = \begin{pmatrix}5\\-6\end{pmatrix} \equiv \overrightarrow{MQ}$

Position vector of $Q = \begin{pmatrix}2\\-5\end{pmatrix} + \begin{pmatrix}5\\-6\end{pmatrix} = \begin{pmatrix}7\\-11\end{pmatrix}$ i.e. $7\mathbf{i} - 11\mathbf{j}$

Line divided into a ratio

E.g. 3 Let $\overrightarrow{OA} = \mathbf{a}$ and let T be on OA such that T divides OA in the ratio 3 : 1.

(a) Is T closer to O or closer to A?

(b) Express $\overrightarrow{OT}$ in terms of $\mathbf{a}$.

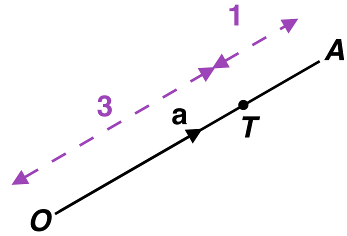
(c) Express $\overrightarrow{AT}$ in terms of $\mathbf{a}$.

Hint: Draw a diagram.

Working: (a) Closer to A

(b) $\overrightarrow{OT} = \dfrac{3}{4}\mathbf{a}$

(c) $\overrightarrow{AT} = -\overrightarrow{TA} = -\dfrac{1}{4}\overrightarrow{OA} = -\dfrac{1}{4}\mathbf{a}$



In general, if T divides OA in the ratio $p : q$ then: $\overrightarrow{OT} = \dfrac{p}{p+q}\mathbf{a}$ $\overrightarrow{TA} = \dfrac{q}{p+q}\mathbf{a}$.

E.g. 4 $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AC} = \mathbf{q}$. The point X lies on BC and divides it in the ratio 2 : 5. Find $\overrightarrow{AX}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

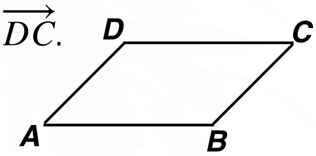
Hint: Draw a diagram.

E.g. 5 Four points have coordinates $A(2, -1)$, $B(k, k+1)$, $C(2k-3, 2k+2)$ and $D(k-1, k)$.

(a) Show that $ABCD$ is a parallelogram for all values of k.

(b) Find the value of k for which $ABCD$ is a rhombus.

Working: (a) For a parallelogram we need to show $\overrightarrow{AB} = \overrightarrow{DC}$.



$\overrightarrow{AB} = \begin{pmatrix} k \\ k+1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} k-2 \\ k+2 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} 2k-3 \\ 2k+2 \end{pmatrix} - \begin{pmatrix} k-1 \\ k \end{pmatrix} = \begin{pmatrix} k-2 \\ k+2 \end{pmatrix}$

Since both $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal to $\begin{pmatrix} k-2 \\ k+2 \end{pmatrix}$, $ABCD$ is a parallelogram.

(b) For $ABCD$ to be a rhombus $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\right|$

$\overrightarrow{AD} = \begin{pmatrix} k-1 \\ k \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} k-3 \\ k+1 \end{pmatrix}$

$\left|\overrightarrow{AB}\right| = \sqrt{(k-2)^2 + (k+2)^2} = \sqrt{2k^2+8}$

$\left|\overrightarrow{AD}\right| = \sqrt{(k-3)^2 + (k+1)^2} = \sqrt{2k^2-4k+10}$

Equating and squaring both sides: $2k^2 + 8 = 2k^2 - 4k + 10$

$4k = 2$

$k = \dfrac{1}{2}$

Video: Vector geometry A
Video: Vector geometry B

Solutions to Starter and E.g.s

Exercise
p242 12D Qu 2i, 3iabc, 4-7, 9-11

Summary

To prove the ***collinearity*** of A, B and C show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel i.e. are multiples of each other. Then state:
"Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel and share the common point A, the points A, B and C must be collinear."

If T divides OA in the ratio $p : q$ then: $\overrightarrow{OT} = \dfrac{p}{p+q}\mathbf{a}$ and $\overrightarrow{TA} = \dfrac{q}{p+q}\mathbf{a}$ where $\mathbf{a} = \overrightarrow{OA}$