Applications of the binomial theorem

Starter

1. *(Review of last lesson)* The first three terms in the expansion of $(A+x)^m$ in ascending powers of x are $64+192x+Bx^2$. Find the values of m, A and B.

Working:

$$(A+x)^m = A^m + {}^mC_1 A^{m-1}x + {}^mC_2 A^{m-2}x^2 + \ldots$$
$$= A^m + mA^{m-1}x + \frac{m(m-1)}{2}A^{m-2}x^2$$

Equating coefficients:

constant: $A^m = 64 \quad \Rightarrow \quad m=3, A=4$ or $m=6, A=2$

Try $A=4, m=3$

x: $mA^{m-1} = 3\times 4^2 = 48 \neq 192$

Try $A=2, m=6$

x: $mA^{m-1} = 6\times 2^6 = 192$ ✓ $\quad \Rightarrow \quad m=6, A=2$

x^2: $B = \frac{m(m-1)}{2}A^{m-2} = \frac{6(6-1)}{2}\times 2^4 = 240$

$m=6, A=2, B=240$

2. (a) Obtain the expansion of $(1+2x)^9$ up to and including the term in x^3, expressing each term in simplified form.

(b) Hence find the first four terms in ascending powers of x in the expansion of $(1-3x)(1+2x)^9$

Working: (a)

$$(1+2x)^9 = 1 + 9\times 2x + {}^9C_2\times(2x)^2 + {}^9C_3\times(2x)^3 + \ldots$$
$$= 1 + 18x + 144x^2 + 672x^3 + \ldots$$

(b)

$$(1-3x)(1+2x)^9 = (1-3x)(1+18x+144x^2+672x^3+\ldots)$$
$$= 1+18x+144x^2+672x^3-3x-54x^2-432x^3$$
$$= 1+15x+90x^2+240x^3$$

E.g. 1 Find the simplified binomial expansion of $(2x+3)(1-2x)^{10}$in ascending powers of x up to and including the term in x^2.

Working:

$$(1-2x)^{10} = 1 + 10\times(-2x) + {}^{10}C_2\times(-2x)^2$$
$$= 1 - 20x + 180x^2 - \ldots$$
$$(2x+3)(1-2x)^{10} = (2x+3)(1-20x+180x^2-\ldots)$$
$$= 2x - 40x^2 + 3 - 60x + 540x^2 + \ldots$$
$$= 3 - 58x + 500x^2$$

E.g. 2 If x is so small that x^3 and higher powers of x are negligible, find the expansion of $(6-11x)(1-4x)^{15}$.

Working:

$$(1-4x)^{15} = 1 + 15\times(-4x) + {}^{15}C_2\times(-4x)^2$$
$$= 1 - 60x + 1680x^2 - \ldots$$
$$(6-11x)(1-4x)^{15} = (6-11x)(1-60x+1680x^2-\ldots)$$
$$= 6 - 360x + 10080x^2 - 11x + 66x^2 - \ldots$$
$$= 6 - 371x + 10146x^2 - \ldots$$

E.g. 3 (a) Obtain the expansion of $\left(x + \frac{2}{x}\right)^4$, expressing each term in simplified form.

(b) Find the coefficient of x^2 in the expansion of $\left(1 + x^2\right)\left(x + \frac{2}{x}\right)^4$

Working: (a) $$\left(x + \frac{2}{x}\right)^4 = x^4 + 4x^3 \times \frac{2}{x} + 6x^2 \times \left(\frac{2}{x}\right)^2 + 4x^2 \times \left(\frac{2}{x}\right)^3 + \left(\frac{2}{x}\right)^4$$
$$= x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}$$

(b) $$\left(1 + x^2\right)\left(x + \frac{2}{x}\right)^4 = \left(1 + x^2\right)\left(x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}\right)$$

Term in x^2 is $8 + 24 = 32$

E.g. 4 Given that that the coefficient of x in the expansion of $(1 + ax)(1 + 5x)^{40}$ is 207, determine the value of a.

Working: $(1 + ax)(1 + 5x)^{40} = (1 + ax)(1 + 200x + \ldots)$

Coefficient of x: $a + 200 = 207 \quad \Rightarrow \quad a = 7$

E.g. 5 Using the first three terms in the expansion of $(1 + x)^6$, find 1.01^6 to 4 decimal places

Working: $(1 + x)^6 = 1 + 6x + {}^6C_2 x^2 + \ldots = 1 + 6x + 15x^2 + \ldots$

If $(1 + x)^6 = 1.01^6 = (1 + 0.01)^6$ then $x = 0.01$

$$1.01^6 = 1 + 6 \times 0.01 + 15 \times 0.01^2 + \ldots$$
$$= 1 + 0.06 + 0.0015$$
$$= 1.0615$$

E.g. 6 Use the first three terms of the expansion of $(1 - 3x)^4$ to find an approximate value for 0.7^4.

Working: $$(1 - 3x)^4 = 1 + 4 \times (-3x) + 6 \times (-3x)^2 + \ldots$$
$$= 1 - 12x + 54x^2 + \ldots$$

If $(1 - 3x)^4 = 0.7^4 = (1 - 0.3)^4 = (1 - 3 \times 0.1)^4$ then $x = 0.1$

$$0.7^4 \approx 1 - 12 \times 0.1 + 54 \times 0.1^2 + \ldots$$
$$= 1 - 1.2 + 0.54$$
$$= 0.34$$

N.B. $0.7^4 = 0.2401$ so 0.34 is not a good approximation — the smaller the value of x the better the approximation

E.g. 7 Find the first three terms in the expansion of $(2+5x)^{12}$ and hence find an approximation for 2.005^{12} to two decimal places.

Working:

$$(2+5x)^{12} = 2^{12} + 12 \times 2^{11} \times 5x + 66 \times 2^{10} \times (5x)^2 + \dots$$
$$= 4096 + 122880x + 1689600x^2 + \dots$$
$$2.005^{12} = (2+5x)^{12} = (2+0.005)^{12} = (2+5\times 0.001)^{12}$$
$$\Rightarrow \quad x = 0.001$$
$$2.005^{12} \approx 4096 + 122880 \times 0.001 + 1689600 \times 0.001^2$$
$$= 4096 + 122.880 + 1.689600$$
$$= 4220.5696$$

Video: **Approximating using binomial expansion**

Exam questions: **Binomial expansion (estimating a value)**
Exam questions: **Binomial expansion**

Solutions to Starter and E.g.s

Exercise
p158 9C Qu 1-8, (9 red)