Binomial distribution

Starter

1. *(Review of last lesson)* The probability distribution of a random variable X is given by:

x	1	2	3	4	5
$P(X = x)$	$2k$	k^2	$3k$	$2k^2$	$\frac{1}{12}$

Prove that there is only one possible value of k, and state its value.

Working: The sum of the probabilities is 1:

$$3k^2 + 5k + \frac{1}{12} = 1$$
$$3k^2 + 5k - \frac{11}{12} = 0$$
$$36k^2 + 60k - 11 = 0$$
$$(6k + 11)(6k - 1) = 0$$
$$k = -\frac{11}{6} \text{ or } k = \frac{1}{6}$$

Since $k > 0$, there is only one value of k and that is $\frac{1}{6}$.

2. A $6-$sided dice is rolled four times.
   (a) Write down the calculation needed to find $P(6, 6', 6', 6')$.
   (b) Write down the calculation needed to find $P(6', 6, 6', 6')$.
   (c) Write down the number of ways of getting one 6 when rolling four dice.
   (d) Write down the coefficients in the expansion of $(x + y)^4$ in terms of ${}^nC_r$. Link your answer to (c) to one of the coefficients.
   (e) Write down the number of ways of getting two 6s when rolling four dice.
   (f) Hence find the probability of getting two 6s when rolling four dice.

Working:

(a) $P(6, 6', 6', 6') = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{1}{6} \times \left(\frac{5}{6}\right)^3$

(b) $P(6', 6, 6', 6') = \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{1}{6} \times \left(\frac{5}{6}\right)^3$

(c) 4 ways

(d) ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$
${}^4C_1 = 4$

(e) ${}^4C_2 = 6$ ways

(f) $P(\text{two 6s}) = {}^4C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = \frac{25}{216}$

E.g. 1 Decide whether the following are examples of binomial distributions.

(a) A coin is flipped eight times and the number of heads obtained is counted.

(b) A four-sided dice is rolled until the number 3 appears.

(c) During a week the number of days when it rains is counted.

Working: (a) Binomially distributed

(b) Not binomially distributed because there is not a fixed number of trials.

(c) Not binomially distributed because the weather on one day affects the weather on the next day so the outcomes of each trial are not independent.

E.g. 2 (a) Write down the coefficients of the terms for the expansion of $(x+y)^5$ in terms of ${}^nC_r$.

(b) A pack of pens contains five pens. The probability that a pen is faulty is 0.25.

(i) Write down the number of ways that one pen in the pack could be faulty. Hence write down the probability of getting one faulty pen.

(ii) Calculate the probability of getting three faulty pens.

(c) Another pack has n pens and the probability that a pen is faulty is p.

(i) Write down the probability of a pen not being faulty.

(ii) Write down a formula in terms of n, C and p for the probability of getting one faulty pen in the pack.

(iii) Write down a similar formula for the probability of getting six faulty pens assuming that $n \geq 6$ in the pack.

(iv) Write down a formula in terms for the probability of getting x faulty pens.

Working: (a) ${}^5C_0$ ${}^5C_1$ ${}^5C_2$ ${}^5C_3$ ${}^5C_4$ ${}^5C_5$

(b) (i) Number of ways is ${}^5C_1 = 5$

$P(\text{one faulty pen}) = {}^5C_1 \times 0.25 \times 0.75^4 = \frac{405}{1024}$

(ii) $P(\text{three faulty pens}) = {}^5C_3 \times 0.25^3 \times 0.75^2 = \frac{45}{512}$

(c) (i) $P(\text{not faulty}) = 1 - p$

(ii) $P(\text{one faulty pen}) = {}^nC_1 \times p \times (1-p)^{n-1}$

(iii) $P(\text{six faulty pens}) = {}^nC_6 \times p^6 \times (1-p)^{n-6}$

(iv) $P(x \text{ faulty pens}) = {}^nC_x \times p^x \times (1-p)^{n-x}$

E.g. 3 The random variable X is such that $X \sim \mathrm{B}(9, 0.6)$. Calculate, giving answers to 3 s.f.,:
(a) $P(X = 2)$ (b) $P(X = 5)$ (c) $P(X \geq 8)$ (d) $P(X \geq 2)$

Working: (a) $P(X = 2) = {}^9C_2 \times 0.6^2 \times 0.4^7 = 0.0212$

(b) $P(X = 5) = {}^9C_5 \times 0.6^5 \times 0.4^4 = 0.251$

(c) $P(X \geq 8) = P(X = 8) + P(X = 9)$
$= {}^9C_8 \times 0.6^8 \times 0.4^1 + {}^9C_9 \times 0.6^9 \times 0.4^0$
$= 0.0705$

(d) $P(X \geq 2) = 1 - P(X < 2)$
$= 1 - P(X = 0) - P(X = 1)$
$= 1 - {}^9C_0 \times 0.6^0 \times 0.4^9 - {}^9C_1 \times 0.6^1 \times 0.4^8$
$= 0.996$

E.g. 4 Given that about $12\,\%$ of the population are left-handed. In a group of eight people, find the probability that:
(a) three people are left-handed
(b) six people are right-handed
(c) less than six people are left-handed.

Working: (a) P(three people are left-handed) $= {}^8C_3 \times 0.12^3 \times 0.88^5$
$= 0.0511$

(b) P(six people are right-handed) $= P$(two people are left-handed)
$= {}^8C_2 \times 0.12^2 \times 0.88^6$
$= 0.187$

(c) $P(<$ six are left-handed$) = 1 - P(\geq$ six are LH$)$
$= 1 - P(X = 6) - P(X = 7) - P(X = 8)$
$= 1 - {}^8C_6 \times 0.12^6 \times 0.88^2 - 8 \times 0.12^7 \times 0.88 - 0.12^8$
$= 0.9999$

E.g. 5 The random variable $X \sim \mathrm{B}(5, p)$ is such that $P(X = 1) = P(X = 2)$. Find the value of p.

Working: $P(X = 1) = P(X = 2)$: ${}^5C_1 \times p \times (1 - p)^4 = {}^5C_2 \times p^2 \times (1 - p)^3$
$5(1 - p) = 10p$
$1 - p = 2p$
$p = \dfrac{1}{3}$

Video: Binomial distribution
Video: Binomial distribution on a calculator
Video: Cumulative probability tables

Solutions to Starter and E.g.s

Exercise
p376 17C Qu 1, 2i, 3, 4i, 5-14, (15-19 red)