Calculating binomial coefficients

Starter

1. ***(Review of last lesson)*** Find the terms indicated in the expansions of the following:

(a) $(4+2p^2)^6$ term in p^6 (b) $\left(4p+\frac{2}{p}\right)^5$ term in p^3

Working: (a) General term of $(4+2p^2)^6$ is ${}^6C_r \times 4^{6-r} \times (2p^2)^r$

Term in p^6 $\Rightarrow$ $2r=6$ $\Rightarrow$ $r=3$

${}^6C_3 \times 4^{6-3} \times (2p^2)^3 = 20 \times 63 \times 8p^6 = 10240p^6$

(b) General term of $\left(4p+\frac{2}{p}\right)^5$ is ${}^5C_r \times (4p)^{5-r} \times \left(\frac{2}{p}\right)^r$

Term in p^3 $\Rightarrow$ $5-r-r=3$ $\Rightarrow$ $r=1$

${}^5C_1 \times (4p)^4 \times \left(\frac{2}{p}\right)^1 = 5 \times 256p^4 \times \frac{2}{p} = 2560p^3$

2. ***(Review of last lesson)*** Expand $(1+x+6x^2)^3$.

Working: $(1+x+6x^2)^3 = \left(1+(x+6x^2)\right)^3$

$= 1+3(x+6x^2)+3(x+6x^2)^2+(x+6x^2)^3$

$= 1+3x+18x^2+3(x^2+12x^3+36x^4)+(x^3+3x^2\times 6x^2+3x\times(6x^2)^2+216x^6)$

$= 1+3x+18x^2+3x^2+36x^3+108x^4+x^3+18x^4+108x^5+216x^6$

$= 1+3x+21x^2+37x^3+126x^4+108x^5+216x^6$

3. Factorial notation, $n!$, is given by:

$$n! = n \times (n-1) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

with $0!$ defined as 1.

State the values of: (a) $4!$ (b) $7!$ (c) $12!$

Working: (a) $4! = 4 \times 3 \times 2 \times 1 = 24$

(b) $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

(c) $12! = 479001600$

4. $^nC_r = \dfrac{n!}{(n-r)!r!}$

(a) Show that $^nC_0 = 1$ and $^nC_n = 1$.

(b) Find expressions in terms of n, for:

(i) $^nC_1$ (ii) $^nC_{n-1}$ (iii) $^nC_2$ (iv) $^nC_{n-r}$

Working: (a) $^nC_0 = \dfrac{n!}{(n-0)!0!} = \dfrac{n!}{n!} = 1$

$^nC_n = \dfrac{n!}{(n-n)!n!} = \dfrac{n!}{n!} = 1$

(b) (i) $^nC_1$ $^nC_1 = \dfrac{n!}{(n-1)!1!} = n$

(ii) $^nC_{n-1} = \dfrac{n!}{(n-(n-1))!(n-1)!} = \dfrac{n!}{1!(n-1)!} = n$

(iii) $^nC_{n-2} = \dfrac{n!}{(n-(n-2))!(n-2)!} = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)}{2}$

(iv) $^nC_{n-r} = \dfrac{n!}{(n-(n-r))!(n-r)!} = \dfrac{n!}{r!(n-r)!} = {}^nC_r$

E.g. 1 Without a calculator, find the value of:

(a) $^5C_2$ (b) $^7C_3$ (c) $^8C_6$ (d) $^9C_3$

Working: (a) $^5C_2 = \dfrac{5!}{(5-2)!2!} = \dfrac{5!}{3!2!} = \dfrac{5\times 4}{2\times 1} = 10$

(b) $^7C_3 = \dfrac{7!}{(7-3)!3!} = \dfrac{7!}{4!3!} = \dfrac{7\times 6\times 5}{3\times 2\times 1} = 35$

(c) $^8C_6 = \dfrac{8!}{(8-2)!6!} = \dfrac{8!}{6!2!} = \dfrac{8\times 7}{2\times 1} = 28$

(d) $^9C_3 = \dfrac{9!}{(9-3)!3!} = \dfrac{9!}{6!3!} = \dfrac{9\times 8\times 7}{3\times 2\times 1} = 84$

E.g. 2 In the expansion of $(1+ax)^4$, the coefficient of x^3 is 1372. Find the value of a.

Working: Term is x^3 is $^4C_3 \times 1^1 \times (ax)^3 = 4a^3x^3$.

So $4a^3 = 1372 \quad \Rightarrow \quad a^3 = 343 \quad \Rightarrow \quad a = 7$

E.g. 3 Given that the expansion of $(1+ax)^n$ begins $1+36x+576x^2$, find the values of a and n.

Working:

$$(1+ax)^n = 1 + {}^nC_1 \times ax + {}^nC_2 \times (ax)^2 + \dots$$
$$= 1 + nax + {}^nC_2 \times a^2x^2 + \dots$$

But ${}^nC_2 = \dfrac{n!}{(n-1)!2!} = \dfrac{n(n-1)}{2}$

So $1+36x+576x^2 \equiv 1 + nax + \dfrac{n(n-1)}{2}a^2x^2 + \dots$

Equating coefficients:

x: $\quad 36 = na$

x^2: $\quad 576 = \dfrac{n(n-1)}{2}a^2$

Substitute $a = \dfrac{36}{n}$: $\quad 576 = \dfrac{n(n-1)}{2} \times \dfrac{36^2}{n^2}$

$$576 = \frac{648(n-1)}{n}$$
$$576n = 648n - 648$$
$$n = 9$$

When $n = 9$, $a = \dfrac{36}{9} = 4$

$a = 4, n = 9$

Video: Binomial expansion

Exam questions: Binomial expansion (positive integer powers)

Exam questions: Comparing coefficients

Solutions to Starter and E.g.s

Exercise

p155 9B Qu 1i, 2i, 3-4, (5-6 red)