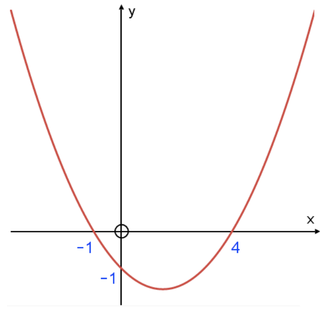
Completing the Square

Starter

1. **(Review of last lesson)**

The sketch is a quadratic function of the form $y = ax^2 + bx + x$. Find the value of a, b and c.



Working:

Root at $x = -1$ so factor is $(x+1)$
Root at $x = 4$ so factor is and $(x-4)$
$y = k(x+1)(x-4)$
When $x = 0$, $y = -1$ so $-1 = k \times 1 \times -4$
i.e. $k = \frac{1}{4}$

Expanding: $y = \frac{1}{4}(x^2 - 3x - 4) = -\frac{1}{4}x^2 - \frac{3}{4}x - 1$

So $a = \frac{1}{4}$, $b = -\frac{3}{4}$ and $c = -1$

2. **(Review of GCSE material)**

(a) Express $x^2 - 2x + 6$ in completed square form i.e. $(x+g)^2 + h$.
(b) State the coordinates of the vertex (or turning point) of $y = x^2 - 2x + 6$.

Working:

(a) $x^2 - 2x + 6 \equiv (x-1)^2 - 1^2 + 6 = (x-1)^2 + 5$

(b) (1, 5)

E.g. 1 Consider the graph of $y = (x-2)^2 + 5$. State:

(a) the coordinates of the vertex
(b) whether the vertex is a maximum or minimum
(c) the equation of the line of symmetry
(d) the greatest or least value, indicating whether it is the greatest or least value

Working:

(a) Vertex is at $(2, 5)$
(b) +ve sign in front of bracket so concave-up ∴ a minimum
(c) Line of symmetry: $x = 2$ ***vertical line through the vertex***
(d) Minimum so 5 is the least value

E.g. 2 For these quadratic curves state (i) the coordinates of the vertex, (ii) the equation of the line of symmetry and (iii) the greatest or least value, indicating whether it is a greatest or least value.

(a) $y = 5(x+4)^2 - 8$ (b) $y = 7 - 3(x-1)^2$
(c) $y = -1 - (x+8)^2$ (d) $y = 6(x-9)^2 + 2$

Working:

(a)
(i) $(-4, -8)$
(ii) $x = -4$ ***vertical line through the vertex***
(iii) $+5x^2$ so minimum: least value is -8

(b)
(i) $(1, 7)$
(ii) $x = 1$ ***vertical line through the vertex***
(iii) $-3x^2$ so maximum: greatest value is 7

(c) (i) $(-8, -1)$
(ii) $x = -8$ *vertical line through the vertex*
(iii) $-x^2$ so maximum: greatest value is -1

(d) (i) $(9, 2)$
(ii) $x = 9$ *vertical line through the vertex*
(iii) $+6x^2$ so minimum: least value is 2

E.g. 3 Express $x^2 - 2x + 10$ in completed square form:

Working: Half of -2 is -1: so $x^2 - 2x + 10 \equiv (x-1)^2 \ldots$
Subtract the square of the number in the bracket: $(x-1)^2 - (-1)^2 \ldots$
Include the constant term at the end: $(x-1)^2 - (-1)^2 + 10$

$$x^2 - 2x + 10 \equiv (x-1)^2 - (-1)^2 + 10$$
$$= (x-1)^2 + 9$$

E.g. 4 Express these quadratic expressions in completed square form:
(a) $x^2 + 4x + 20$ (b) $x^2 - 9x - 10$

Working: (a) $x^2 + 4x + 20 \equiv (x+2)^2 - 2^2 + 20 = (x+2)^2 + 16$

(b) $x^2 - 9x - 10 \equiv (x-4.5)^2 - (-4.5)^2 - 10 = (x-4.5)^2 - 30.25$

E.g. 5 Express these quadratic expressions in completed square form:
(a) $2x^2 - 16x + 9$ (b) $1 - 2x - x^2$

Working: (a)

$$2x^2 - 16x + 9 \equiv 2\left[x^2 - 8x\right] + 9$$ *factorise the coefficient of x^2*
$$= 2\left[(x-4)^2 - (-4)^2\right] + 9$$ *complete the square*
$$= 2\left[(x-4)^2 - 16\right] + 9$$
$$= 2(x-4)^2 - 32 + 9$$ *expand square brackets*
$$= 2(x-4)^2 - 23$$

(b)

$$1 - 2x - x^2 \equiv 1 - \left[x^2 + 2x\right]$$ *factorise the coefficient of x^2*
$$= 1 - \left[(x+1)^2 - 1^2\right]$$ *complete the square*
$$= 1 - \left[(x+1)^2 - 1\right]$$
$$= 1 - (x+1)^2 + 1$$ *expand square brackets*
$$= 2 - (x+1)^2$$

E.g. 6 For these quadratic expressions (i) express them in completed square form (ii) state the coordinates of the turning point:
(a) $y = 2x^2 + 12x + 5$ (b) $y = 5 + 10x - x^2$

Working: (a) (i)

$$2x^2 + 12x + 5 \equiv 2\left[x^2 + 6x\right] + 5$$
$$= 2\left[(x+3)^2 - 3^2\right] + 5$$
$$= 2(x+3)^2 - 18 + 5$$
$$= 2(x-4)^2 - 13$$
$$y = 2(x+3)^2 - 13$$

(ii) $(-3, -13)$

(b) (i) $5 + 10x - x^2 \equiv 5 - [x^2 - 10x]$
$= 5 - [(x-5)^2 - (-5)^2]$
$= 5 - [(x-5)^2 - 25]$
$= 30 - (x-5)^2$
$y = 30 - (x-5)^2$

(ii) (5, 30)

E.g. 7 Find the equation of the quadratic in the form $y = ax^2 + bx + c$ given that the vertex is at (−3, 12) and the curve passes through the point (1, −4).

Working: Vertex is at (−3, 12) so $y = 12 - k(x+3)^2$
Curve passes through the point (1, −4): $-4 = 12 - k(1+3)^2$
$-4 = 12 - 16k$
So $k = 1$
Expanding gives: $y = 3 - 6x - x^2$

E.g. 8 Given that $g(x) = x^2 + 8x + 20$, show that $g(x) \geq 4$ for all values of x.

Working: $g(x) = x^2 + 8x + 20 \equiv (x+4)^2 - 16 + 20 = (x+4)^2 + 4$
Since $(x+4)^2 \geq 0$ then $(x+4)^2 + 4 \geq 4$

E.g. 9 Solve the equation $x^2 - 7x - 1 = 0$ by completing the square. Give your answers exactly.

Working: $x^2 - 7x - 1 \equiv (x-3.5)^2 - (-3.5)^2 - 1 = (x-3.5)^2 - 13.25$

$$\left(x - \frac{7}{2}\right)^2 - \frac{53}{4} = 0$$

$$\left(x - \frac{7}{2}\right)^2 = \frac{53}{4}$$

$$x - \frac{7}{2} = \pm\sqrt{\frac{53}{4}}$$ ***remember the*** $\pm$

$$x = \frac{7}{2} \pm \sqrt{\frac{53}{4}}$$

Video: Completing the square
Video: Applications of completing the square

Solutions to Starter and E.g.s

Exercise
p40 3C Qu 1-3 (i), 4-9, (10)