Discriminant revisited

Starter

1. *(Review of last lesson)* Find the points of intersection between the line $x + 2y = 3$ and the curve $x^2 + xy = 2$.

Working: $x + 2y = 3 \quad \Rightarrow \quad x = 3 - 2y$

Substitute: $(3 - 2y)^2 + y(3 - 2y) = 2$

Expand: $4y^2 - 12y + 9 + 3y - 2y^2 = 2$

$$2y^2 - 9y + 7 = 0$$
$$(2y - 7)(y - 1) = 0$$
$$y = \frac{7}{2} \quad \text{or} \quad y = 1$$

When $y = \frac{7}{2}$, $x = 3 - 2 \times \frac{7}{2} = -4$

When $y = 1$, $x = 3 - 2 \times 1 = 1$

The points of intersection are at $\left(-4, \frac{7}{2}\right)$ and $(1, 1)$.

2. Find the ***number*** of points of intersection between the line $4y - x = 16$ and the curve $y^2 = 4x$. What can be said about the line and the curve?

Worked: $y - x = 4 \quad \Rightarrow \quad y = x + 4$

Substitute into $y^2 - 5x^2 = 20$: $(x + 4)^2 - 5x^2 = 20$

Expand and rearrange: $x^2 + 8x + 16 - 5x^2 = 20$

$$4x^2 - 8x + 4 = 0$$
$$x^2 - 2x + 1 = 0$$

Using the discriminant $b^2 - 4ac$ where $a = 1, b = -2, c = 1$

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 1 = 0$$

Since the discriminant is zero, the line and curve have one point of intersection i.e. the line is a ***tangent*** to the curve.

E.g. 1 Show algebraically that the line $y = 3x - 3$ and the curve $y = (3x + 1)(x + 2)$ do not meet.

Working: Substitute $y = 3x - 3$: $(3x + 1)(x + 2) = 3x - 3$

$$3x^2 + 3x + 2 = 3x - 3$$
$$3x^2 = -5$$

No real solution since we cannot square root a negative number. Therefore, the line and the curve do not meet.

E.g. 2 Find the value(s) of k for which the line $y - x = k$ is tangent to the curve $y^2 - 5x^2 = 20$.

Worked: $y - x = k \quad \Rightarrow \quad y = x + k$

Substitute into $y^2 - 5x^2 = 20$***:*** $(x+k)^2 - 5x^2 = 20$

Expand: $x^2 + 2kx + k^2 - 5x^2 = 20$

$4x^2 - 2kx + 20 - k^2 = 0$

If the line is a tangent $b^2 - 4ac = 0$: $a = 4, b = -2k, c = 20 - k^2$

$(-2k)^2 - 4 \times 4 \times (20 - k^2) = 0$

$4k^2 + 16k^2 - 320 = 0$

$k^2 = 16$

Therefore, $k = \pm 4$

E.g. 3 Find the range of values of m for which the line $y = mx + 1$ intersects the curve $y = x^2 - x + 3$ in two distinct places. Give your answers exactly.

Working: ***The curve and line intersect when:*** $x^2 - x + 3 = mx + 1$

$x^2 - (m+1)x + 2 = 0$

"two distinct places" $\Rightarrow$ $b^2 - 4ac > 0$

$a = 1 \qquad b = -(m+1) \qquad c = 2$

$[-(m+1)]^2 - 4 \times 1 \times 2 > 0$

$m^2 + 2m - 7 > 0$

Roots are: $m = -1 + 2\sqrt{2}$ and $m = -1 - 2\sqrt{2}$

$> 0 \quad \Rightarrow$ above the $m-$axis

Since the curve $y = m^2 + 2m - 7$ is concave-up

$m < -1 - 2\sqrt{2}$ and $m > -1 + 2\sqrt{2}$

Video: Nature of the intersection

Solutions to Starter and E.g.s

Exercise

p73 5B Qu 1-3, (4-5 red)