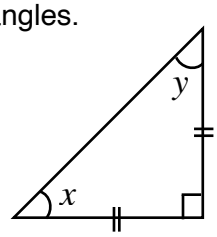
Exact values of trigonometric functions

Starter

1. ***(Review of last lesson)*** Consider the right-angled isosceles triangle with angles.



(a) State the size of angles x and y.

(b) Given that the length of the opposite and adjacent sides are 1 unit, calculate the length of the hypotenuse.

(c) Hence write down the values of $\tan 45°$, $\sin 45°$ and $\cos 45°$. Rationalise the denominator where necessary.

Working:

(a) $x = y = 45°$

(b) Let the hypotenuse side be a.

By Pythagoras' theorem: $a^2 = 1^2 + 1^2$

$$a^2 = 2$$

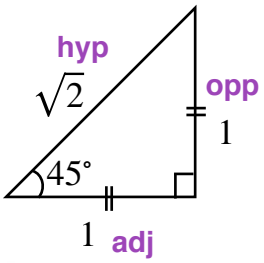
$$a = \sqrt{2}$$

The length of the hypotenuse is $\sqrt{2}$.

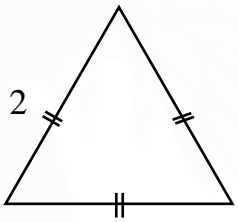
(c) $$\tan 45° = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

$$\sin 45° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$



2. Draw an equilateral triangle of side 2 units.



(a) Calculate the height of the triangle.

(b) Hence write down the values of:

(i) $\tan 60°$, $\sin 60°$ and $\cos 60°$

(ii) $\tan 30°$, $\sin 30°$ and $\cos 30°$

Rationalise the denominator where necessary.

Working:

(a) Let h be the height of the triangle.

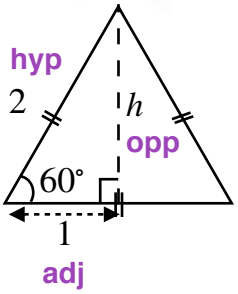
By Pythagoras: $h^2 + 1^2 = 2^2$

$$h^2 + 1 = 4$$

$$h^2 = 3$$

$$h = \sqrt{3}$$

The height of the triangle is $\sqrt{3}$.



(b) (i) $$\tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$$

(ii) $\tan 30^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$

$\cos 30^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}$

Notes

E.g. 1 Collate the values from the starter in the table below and using your knowledge of the trigonometric graphs, complete the rest of the table:

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin									
cos									
tan									

Working:

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

N.B. These values are not given on the formula page so you will need to learn them.

E.g. 2 Using your knowledge of trigonometric graphs, and without using a calculator, find the exact value of:

(a) $\sin 390^\circ$ (b) $\cos 240^\circ$ (c) $\tan 315^\circ$

Working: (a) $\sin 390^\circ = \sin(390^\circ - 360^\circ) = \sin 30^\circ = \frac{1}{2}$

(b) $\cos 240^\circ = \cos(240^\circ - 360^\circ) = \cos(-120^\circ) = \cos 120^\circ = -\frac{1}{2}$

(c) $\tan 315^\circ = \tan(315^\circ - 180^\circ - 180^\circ) = \tan(-45^\circ) = -\tan 45^\circ = -1$

E.g. 3 Without using a calculator, find the value of:

(a) $\cos^2 45° + \sin^2 45°$ (b) $2 \sin 150° \cos 150°$

Working: (a) $\cos^2 45° + \sin^2 45° = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1$

(b) $2 \sin 150° \cos 150° = 2 \times \frac{1}{2} \times -\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

E.g. 4 Find the value of θ, where $0° \leq \theta \leq 90°$, such that:

(a) $\sin\theta \cos 315° + \sin 45° \cos 30° = \frac{\sqrt{6}+\sqrt{2}}{4}$

(b) $\cos^2\theta + \sin^2 240° = 1$

(c) $\tan\theta = \frac{2\tan 45°}{1 - \tan^2 45°}$

Working: (a) $\cos 315° = \cos(315° - 360°) = \cos(-45°) = \cos 45° = \frac{\sqrt{2}}{2}$

$$\sin\theta \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Multiply by 4: $\sin\theta \times 2\sqrt{2} + \sqrt{6} = \sqrt{6} + \sqrt{2}$

$$\sin\theta = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

$\therefore \theta = 30°$

(b) $\sin 240° = \sin(240° - 360°) = \sin(-120°) = -\sin 120° = -\sin 60° = -\frac{\sqrt{3}}{2}$

$$\cos^2\theta + \sin^2 240° = 1 \quad \Rightarrow \quad \cos^2\theta + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \pm\frac{1}{2}$$

Since $0° \leq \theta \leq 90°$, $\cos\theta = \frac{1}{2}$

$\therefore \theta = 60°$

(c) $\frac{2\tan 45°}{1 - \tan^2 45°} = \frac{2 \times 1}{1 - 1^2} = \frac{2}{0}$ so $\tan\theta$ is undefined

$\therefore \theta = 90°$

Video: Trigonometric ratios for 30, 45 and 60 degrees

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Solutions to Starter and E.g.s

Exercise

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