Exponential Modelling

Starter

1. **(Review of a previous lesson)**

Solve, giving your answers exactly: (a) $e^{x-3} = 14$ (b) $5e^{3x} = 45$.

N.B. For (b), "Take ln of both sides" since the equation includes e.

Working: (a) Take ln of both sides: $\ln e^{x-3} = \ln 14$
3rd law: $(x-3)\ln e = \ln 14$
Since $\ln e = 1$: $x - 3 = \ln 14$
$x = 3 + \ln 14$

N.B. "Take ln of both sides" since the equation includes e.

(b) Divide by 5: $e^{3x} = 9$
Take ln of both sides: $\ln e^{3x} = \ln 9$
3rd law: $3x \ln e = \ln 9$
Since $\ln e = 1$: $3x = \ln 9$
$x = \frac{1}{3}\ln 9$

E.g. 1 The concentration (C) of a drug in the bloodstream, t hours after taking an initial dose, decreases exponentially according to $C = Ae^{-kt}$, where A and k are constants.

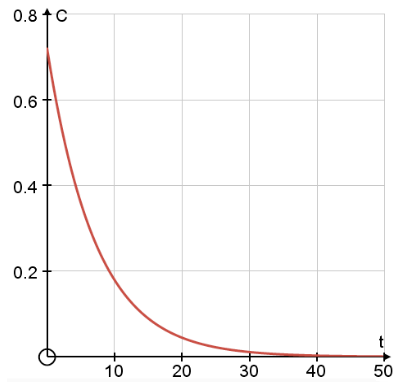
(a) If the initial concentration is 0.72, and this halves after 5 hours, find the values of A and k.
(b) Find the rate of change of the concentration when $t = 2$
(c) Sketch the graph of C against t.

Working: (a) Initial concentration is 0.72 so when $t = 0$, $C = 0.72$
$72 = Ae^{-k\times 0}$ $\Rightarrow$ $A = 0.72$ since $e^0 = 1$

When $t = 5$, $C = \frac{0.72}{2} = 0.36$
$0.36 = 0.72e^{-5k}$ $\Rightarrow$ $0.5 = e^{-5k}$
$\ln 0.5 = \ln e^{-5k}$ $\Rightarrow$ $\ln 0.5 = -5k$ *since* $\ln e = 1$
$-\frac{1}{5}\ln 0.5 = k$ $\Rightarrow$ $k = 0.139$ (3 s.f.)
$C = 0.72e^{-0.139t}$

(b) Rate of change of concentration is given by $-0.139 \times 0.72e^{-0.139t}$
When $t = 2$, rate of change is $-0.139 \times 0.72e^{-0.139\times 2}$
So rate of change is -0.0758.

(c) $t \geq 0$
When $t = 0$, $C = 0.72$
Exponentially decay



E.g. 2 The exponential growth of a colony of bacteria can be modelled by the equation $B = 60e^{0.03t}$, where B is the number of bacteria and t is the time in hours from the point the colony is first monitored ($t \geq 0$). Use the model to:
(a) Work out the initial population of bacteria.
(b) Predict the number of bacteria after 4 hours.
(c) The growth of a different bacteria is modelled by the function $25e^{0.1t}$. Compare the two population models.

Working: (a) When $t = 0, B = 60$ ***since*** $e^0 = 1$

(b) When $t = 4, B = 67$ $B = 67.65$ ***but no 0.65 of bacteria***

(c) The initial bacteria of the 2nd model is 25 so it has a smaller population at the start.
However, since the coefficient of t is 0.1 which is bigger than 0.03, the 2nd population grows at a faster rate.

E.g. 3 £350 is initially paid into a bank account that pays 3% interest per year. No further money is deposited or withdrawn from the account. Write down an equation to show how much money will be in the account after t years. Use your model to calculate how many whole years it will take before there is more than £1000 in the account.

Working: Adding 3% each year $\Rightarrow$ $\times 1.03$
Money in the account after t years $= 350 \times 1.03^t$
$350 \times 1.03^t > 1000$
$$1.03^t > \frac{1000}{350}$$
Take logs of both sides:
$$\log 1.03^t > \log \frac{100}{35}$$
$$t \log 1.03 > \log \frac{20}{7}$$
$$t > \frac{\log 20 - \log 7}{\log 1.03}$$
N.B. Since $\log 1.03 > 0$, the inequality sign does not change direction
$$t > 35.5$$
It will take 36 years to reach £1000 in the account.

E.g. 4 The penguin population, P, of a small island can be modelled by the formula $P = 5000e^{0.1t}$, where t is the number of years after the initial survey.
(a) What does 5000 represent in the formula.
(b) Explain why this model may not be appropriate for the long term.

Working: (a) 5000 is the initial number of penguins i.e. when $t = 0$.

(b) The model suggests the population will tend to infinity
(e.g. after 60 years the populations will be over 2 million).
This is unrealistic as it does not take into other factors such as finite food supply, predators or possible disease from overcrowding.

Video: Exponential growth problem
Video: Exponential growth and decay problems

Solutions to Starter and E.g.s

Exercise
p139 8C Qu 2-7, 9 (10-12)