Geometrical Significance of Definite Integration

Starter

1. **(Review of last lesson)** Let $\int_1^4 f(x)dx = 8$. Find $\int_1^4 (5f(x) + 2x + 3)dx$.

Working:

$$\begin{aligned}\int_1^4 (5f(x) + 2x + 3)dx &= \int_1^4 5f(x)dx + \int_1^4 (2x+3)dx \\ &= 5\int_1^4 f(x)dx + \left[x^2 + 3x)\right]_1^4 \\ &= 5 \times 8 + \left(4^2 + 3 \times 4\right) - \left(1^2 + 3 \times 1\right) \\ &= 40 + 16 + 12 - 1 - 3 \\ &= 64\end{aligned}$$

2. **(Review of last lesson)**

Find the two possible values for A that satisfy $\int_{-2}^2 \left(\frac{21}{8}x^2 + \frac{A}{x^2}\right)dx = 3A^2$.

Working:

$$\begin{aligned}\int_{-2}^2 \left(\frac{21}{8}x^2 + \frac{A}{x^2}\right)dx &= \int_{-2}^2 \left(\frac{21}{8}x^2 + Ax^{-2}\right)dx \\ &= \left[\frac{7}{8}x^3 - Ax^{-1}\right]_{-2}^2 \\ &= \left(\frac{7}{8} \times 2^3 - \frac{A}{2}\right) - \left(\frac{7}{8} \times (-2)^3 - \frac{A}{-2}\right) \\ &= \left(7 - \frac{A}{2}\right) - \left(-7 + \frac{A}{2}\right) \\ &= 14 - A\end{aligned}$$

So $3A^2 = 14 - A \quad \Rightarrow \quad 3A^2 + A - 14 = 0$

$\Rightarrow \quad (3A + 7)(A - 2) = 0 \qquad \therefore A = -\frac{7}{3}$ or $A = 2$

E.g. 1 Find the area between the curve $y = x^2 + 3$, the x-axis and the lines $x = 1$ and $x = 2$.

Working:

$$\begin{aligned}\int_1^2 (x^2 + 3)dx &= \left[\frac{1}{3}x^3 + 3x\right]_1^2 \\ &= \left(\frac{1}{3} \times 2^3 + 3 \times 2\right) - \left(\frac{1}{3} \times 1^3 + 3 \times 1\right) \\ &= \frac{16}{3} = 5\frac{1}{3}\end{aligned}$$

E.g. 2 Find the area between the curve $y = \sqrt{x}$, the $x-$axis and the lines $x = 4$ and $x = 9$.

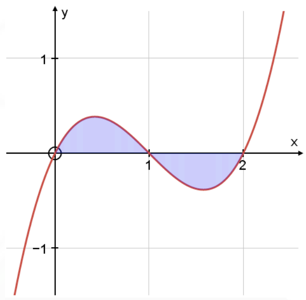
Working:

$$\int_4^9 \sqrt{x}dx = \int_4^9 x^{\frac{1}{2}}dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_4^9$$
$$= \left(\frac{2}{3} \times 9^{\frac{3}{2}}\right) - \left(\frac{2}{3} \times 4^{\frac{3}{2}}\right)$$
$$= 18 - \frac{16}{3}$$
$$= 12\frac{2}{3}$$

E.g. 3 Find the area between the curve $y = x^3 - 3x^2 + 2x$, the $x-$axis for $x = 0$ and $x = 2$. Hence, or otherwise, sketch the graph of $y = x^3 - 3x^2 + 2x$.

Working:

$$\int_0^2 x^3 - 3x^2 + 2x = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^2$$
$$= \left(\frac{1}{4} \times 2^4 - 2^3 + 2^2\right) - \left(0\right) = 0$$



The answer of zero causes some students to believe the curve must run along the $x-$axis between 0 and 2.

In fact, as can be seen from the diagram, the areas above and below the $x-$axis are equal and cancel each other out since ***integrations*** where the function is ***below the $x-$axis*** come out as ***negative***.

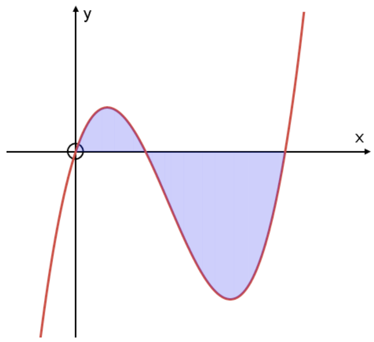
Therefore to get the answer, we do separate integrations:

$$\int_0^1 (x^3 - 3x^2 + 2x)dx = \frac{1}{4} \quad \text{and} \quad \int_1^2 (x^3 - 3x^2 + 2x)dx = -\frac{1}{4}$$

Since area only takes positive values, the negative value is made positive before adding:

Required area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

E.g. 4 Calculate the exact value of the shaded area shown for the curve $y = x^3 - 4x^2 + 3x$.



Working: Solving $x^3 - 4x^2 + 3x = 0$ to find the roots

$x(x^2 - 4x + 3) = 0$

$x(x-1)(x-3) = 0$

The roots are 0, 1 and 3.

Shaded area $= \displaystyle\int_0^1 (x^3 - 4x^2 + 3x)dx - \int_1^3 (x^3 - 4x^2 + 3x)dx$

$$\int_0^1 (x^3 - 4x^2 + 3x)dx = \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0 - 0 + 0) = \frac{5}{12}$$

$$\int_1^3 (x^3 - 4x^2 + 3x)dx = \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$$

$$= \left(\frac{1}{4} \times 3^4 - \frac{4}{3} \times 3^3 + \frac{3}{2} \times 3^2\right) - \frac{5}{12}$$

$$= \frac{81}{4} - 36 + \frac{27}{2} - \frac{5}{12}$$

$$= -\frac{8}{3}$$

So the shaded area $= \dfrac{5}{12} - -\dfrac{8}{3} = \dfrac{37}{12}$

E.g. 5 Find the area enclosed by the curve with equation $y = 3x^2 + 6x - 9$, the $x-$axis and the lines $x = -2$ and $x = 2$.

Working: $y = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x-1)(x+3)$

So the roots are $x = -3$ and $x = 1$

The two areas are $\displaystyle\int_{-2}^{1} (3x^2 + 6x - 9)dx$ and $\displaystyle\int_1^2 (3x^2 + 6x - 9)dx$.

I'll calculate them separately and if the answer is negative, I will simply make it positive.

$$\int_{-2}^{1} (3x^2 + 6x - 9)dx = \left[x^3 + 3x^2 - 9x\right]_{-2}^{1}$$

$$= (1 + 3 - 9) - \left((-2)^3 + 3 \times (-2)^2 - 9 \times (-2)\right)$$

$$= -5 + 8 - 12 - 18 = -27$$

Since it is negative, I need to make it positive.

$$\int_1^2 (3x^2 + 6x - 9)dx = \left[x^3 + 3x^2 - 9x\right]_1^2$$

$$= \left(2^3 + 3 \times 2^2 - 9 \times 2\right) - (1 + 3 - 9)$$

$$= 8 + 12 - 18 + 5 = 7$$

The required area $= 27 + 7 = 34$

Video: Area bound by curve and x-axis

Area bound by curve and x-axis EQ

Solutions to Starter and E.g.s

Exercise
p306 15E Qu 1i, 2i, 3-10