Mathematical structures and arguments

Starter

1. *(Review of last lesson)* Solve $9\cos^2\theta = 5\sin\theta\cos\theta$ for $-180^\circ \le \theta \le 180^\circ$.

Working: $9\cos^2\theta = 5\sin\theta\cos\theta \quad \Rightarrow \quad 9\cos^2\theta - 5\sin\theta\cos\theta = 0$

$\cos\theta(9\cos\theta - 5\sin\theta) = 0$

$\cos\theta = 0$ or $9\cos\theta = 5\sin\theta$

From the graph, $\cos\theta = 0 \quad \Rightarrow \quad \theta = 0$

$9\cos\theta = 5\sin\theta \quad \Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{9}{5} \quad \Rightarrow \quad \tan\theta = \dfrac{9}{5} = 1.8$

$\tan^{-1} 1.8 = 60.9^\circ$

tan is +ve in the **A** & **T** quadrants

Draw the angle from the horizontal

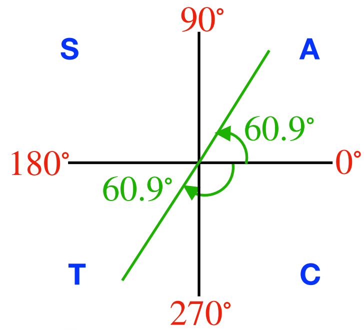
Measure anti-clockwise for A quadrant

Measure clockwise for T quadrant

$\theta = 60.9^\circ$ or $-180^\circ + 60.9^\circ$

$\theta = 60.9^\circ$ or -119°

The required angles are $-119^\circ, 0^\circ, 60.9^\circ$



2. *(Review of last lesson)* Solve $3\sin^2\theta = 4\cos\theta$ for $0^\circ \le \theta \le 360^\circ$.

Working: $3\sin^2\theta = 4\cos\theta \quad \Rightarrow \quad 3(1-\cos^2\theta) = 4\cos\theta$

$3\cos^2\theta + 4\cos\theta - 3 = 0$

$$\cos\theta = \frac{-4 \pm \sqrt{4^2 - 4\times 3\times(-3)}}{2\times 3}$$

$$\cos\theta = \frac{-2 \pm \sqrt{13}}{3}$$

Since $\dfrac{-2-\sqrt{13}}{3} < -1$, $\cos\theta = \dfrac{-2-\sqrt{13}}{3}$ has no solution.

$$\cos\theta = \frac{-2+\sqrt{13}}{3}$$

$$\cos^{-1}\left(\frac{-2+\sqrt{13}}{3}\right) \approx 57.6^\circ$$

$\dfrac{-2+\sqrt{13}}{3}$ is +ve

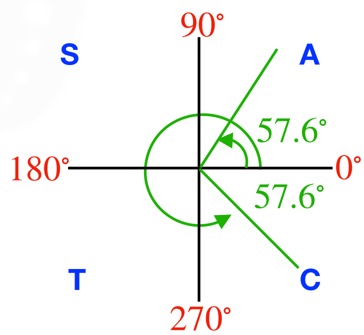
cos is +ve in the **A** & **C** quadrants

Draw the angle from the horizontal

Measure anti-clockwise from the +ve x−axis

$\theta = 57.6^\circ$ or $360^\circ - 57.6^\circ$

$\theta = 57.6^\circ$ or 302°



E.g. 1 Insert either a $\Leftrightarrow$, a $\Rightarrow$ or a $\Leftarrow$ between these statements:

(a) $4x - 5 = 15$ $\square$ $4x = 20$
(b) $x^3 - 1 = 7$ $\square$ $x = 2$
(c) $x^2 - 5 = 11$ $\square$ $x = 4$

Working: (a) $4x - 5 = 15 \Leftrightarrow 4x = 20$

(b) $x^3 - 1 = 7 \Leftrightarrow x = 2$

(c) $x^2 - 5 = 11 \Leftarrow x = 4$

E.g. 2 Use the symbols $\Leftrightarrow$ and $\Rightarrow$ to connect these statements:

A: $\triangle PQR$ is isosceles
B: In $\triangle PQR$, $\angle P = \angle Q$
C: $\triangle PQR$ is equilateral
D: In $\triangle PQR$, $PR = QR$

Give a reason for each connection.

Working:

$B \Rightarrow A$ a triangle with two equal sides is isosceles
$B \Rightarrow D$ (sketch a diagram)
$C \Rightarrow A$ an equilateral triangle is a type of isosceles triangle
$C \Rightarrow B$ an equilateral triangle has all angles equal
$C \Rightarrow D$ an equilateral triangle has all sides equal
$D \Rightarrow A$ two equal sides is the definition of an isosceles triangle
$D \Rightarrow B$ (sketch a diagram)
Since $B \Rightarrow D$ and $D \Rightarrow B$, $B \Leftrightarrow D$

E.g. 3 Use the symbols $\Leftrightarrow$ and $\Rightarrow$ to connect the statements A, B, and C.

(a) A: $PQRS$ is a square
B: $PQRS$ has four equal sides
C: $PQRS$ is a parallelogram

(b) A: $WXYZ$ is a rectangle 3 cm by 4 cm
B: The area of rectangle $WXYZ$ is 12 cm²
C: $WXYZ$ is a rectangle with diagonals 5 cm long

Working: (a) $A \Rightarrow B$
$A \Rightarrow C$
$B \Rightarrow C$

(b) $A \Rightarrow B$
$A \Rightarrow C$

E.g. 4 Use the symbol $\nRightarrow$ to connect these statements:

(a) A: $PQRS$ is a square
B: $PQRS$ has four equal sides
C: $PQRS$ is a parallelogram

(b) A: $WXYZ$ is a rectangle 3 cm by 4 cm
B: The area of rectangle $WXYZ$ is 12 cm^2
C: $WXYZ$ is a rectangle with diagonals 5 cm long

Working: (a) $B \nRightarrow A$
$C \nRightarrow A$
$C \nRightarrow B$

(b) $B \nRightarrow A$
$B \nRightarrow C$
$C \nRightarrow A$
$C \nRightarrow B$

E.g. 5 A student was asked to solve the equation $x + 2\sqrt{x} - 35 = 0$. Here is their working:

Line 1: Let $u = \sqrt{x}$: $u^2 + 2u - 35 = 0$
Line 2: $(u-5)(u+7) = 0$
Line 3: $u = 5$ or $u = -7$
Line 4: $\sqrt{a} = 5$ or $\sqrt{a} = -7$
Line 5: $a = 25$ or $a = 49$

Find the error in the student's working and give the correct solution.

Working: The error is from line 4 to line 5.
While $\sqrt{a} = -7 \Rightarrow a = 49$, there is no real number such that $\sqrt{a} = -7$

N.B. $X \Rightarrow Y$ means ***if*** X is true then Y is true, it does ***not*** state that X ***is*** true.

E.g. 6 (a) A student was asked to solve the equation $\sqrt{2x+1} = \sqrt{x} - 5$. Insert the correct symbol, either a ⇔ or a ⇒, at the start of each line of working. The last one has been done for you.

$\sqrt{2x+1} = \sqrt{x} - 5$

Line 1: $2x + 1 = (\sqrt{x} - 5)^2$
Line 2: $2x + 1 = x - 10\sqrt{x} + 25$
Line 3: $10\sqrt{x} = 24 - x$
Line 4: $100x = (24 - x)^2$
Line 5: $100x = 576 - 48x + x^2$
Line 6: $x^2 - 148x + 576 = 0$
Line 7: $(x-4)(x-144) = 0$
Line 8: ⇔ either $x = 4$ or $x = 144$

(b) Show that the values of $x = 4$ and $x = 144$ are not solutions of the equation $\sqrt{2x+1} = \sqrt{x} - 5$. Explain why.

(c) State the four equations which when squared give $2x + 1 = (\sqrt{x} - 5)^2$.

Working: (a)

$\sqrt{2x+1} = \sqrt{x} - 5$

Line 1: ⇒ $2x + 1 = (\sqrt{x} - 5)^2$
Line 2: ⇔ $2x + 1 = x - 10\sqrt{x} + 25$
Line 3: ⇔ $10\sqrt{x} = 24 - x$
Line 4: ⇒ $100x = (24 - x)^2$
Line 5: ⇔ $100x = 576 - 48x + x^2$
Line 6: ⇔ $x^2 - 148x + 576 = 0$
Line 7: ⇔ $(x-4)(x-144) = 0$
Line 8: ⇔ either $x = 4$ or $x = 144$

(b) When $x = 4$: $\sqrt{2\times 4+1} = 3 \neq \sqrt{4} - 5 = -3$
When $x = 144$: $\sqrt{2\times 144+1} = 17 \neq \sqrt{144} - 5 = 7$
The reasons is that not every step is reversible.

(c) The four equations are:

$\sqrt{2x+1} = \sqrt{x} - 5$
$-\sqrt{2x+1} = \sqrt{x} - 5$
$\sqrt{2x+1} = -\sqrt{x} - 5$
and $-\sqrt{2x+1} = -\sqrt{x} - 5$.

All of them square to get $2x + 1 = (\sqrt{x} - 5)^2$.

Video: Proving identities

Solutions to Starter and E.g.s

Exercise
p5 1A Qu 1i, 2i, 3-10, (11 red)