Nature of Stationary Points

Starter

1. (Review of last lesson) The curve given by $f(x) = x^3 + ax^2 + bx + c$ has a stationary point at $(3, 10)$. Given also that $f''(3) = 0$, find a, b and c.

Working: Curve passes through (3, 10): $3^3 + a \times 3^2 + 3b + c = 10$

$9a + 3b + c = -17$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 0$: $3 \times 3^2 + 2a \times 3 + b = 0 \Rightarrow 6a + b = -27$

$f''(x) = 6x + 2a$

$f''(3) = 0$: $6 \times 3 + 2a = 0 \Rightarrow a = -9$

When $a = -9$: $6 \times (-9) + b = -27 \Rightarrow b = 27$

When $a = -9$ and $b = 27$: $9 \times (-9) + 3 \times 27 + c = -17$

So $c = -17$

$a = -9, b = 27, c = -17$

E.g. 1 The stationary points of the curve $y = x^3 - 15x^2 + 48x + 7$ are (2, 51) and (8, −57).

(a) Use the second derivative method to determine the nature of the stationary point $(2, -51)$.

(b) Use the gradient change method to determine the nature of the stationary point $(8, -57)$.

Working: (a)

$$\frac{dy}{dx} = 3x^2 - 30x + 48$$

$$\frac{d^2y}{dx^2} = 6x - 30$$

When $x = 2$, $\frac{d^2y}{dx^2} = 6 \times 2 - 30 < 0$

$\therefore (2, -51)$ is a maximum

(b)

$$\frac{dy}{dx} = 3x^2 - 30x + 48$$

When $x = 7.9$, $\frac{dy}{dx} = 3 \times 7.9^2 - 30 \times 7.9 + 48 < 0$

When $x = 8.1$, $\frac{dy}{dx} = 3 \times 8.1^2 - 30 \times 8.1 + 48 > 0$

Gradient change: −ve / 0 / +ve

$\therefore (8, -57)$ is a minimum

E.g. 2 Find and classify the stationary points of these curves:

(a) $y = 4x^3 + 3x^2 - 6x - 1$

(b) $y = 3x^4 - 8x^3 + 6x^2 - 3$

(c) $y = x^2 + \frac{16}{x^2}$

Working: (a) $\frac{dy}{dx} = 12x^2 + 6x - 6$

A stationary point occurs when $\frac{dy}{dx} = 0$ so $12x^2 + 6x - 6 = 0$

i.e. $2x^2 + x - 1 = 0$ $\Rightarrow$ $(2x - 1)(x + 1) = 0$

$x = \frac{1}{2}$ or $x = -1$

$\frac{d^2y}{dx^2} = 24x + 6$

When $x = \frac{1}{2}, \frac{d^2y}{dx^2} = 24 \times \frac{1}{2} + 6 > 0$ ***number not required***

So $x = \frac{1}{2}$ gives a maximum

When $x = -1, \frac{d^2y}{dx^2} = 24 \times (-1) + 6 < 0$

So $x = -1$ gives a minimum

When $x = \frac{1}{2}, y = 4 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^2 - 6 \times \left(\frac{1}{2}\right) - 1 = -2\frac{3}{4}$

When $x = -1, y = 4 \times (-1)^3 + 3 \times (-1)^2 - 6 \times (-1) - 1 = 4$

$\left(\frac{1}{2}, -2\frac{3}{4}\right)$ is a minimum and $(-1, 4)$ is a maximum

(b) $\frac{dy}{dx} = 12x^3 - 24x^2 + 12x$

A SP occurs when $\frac{dy}{dx} = 0$ so $12x^3 - 24x^2 + 12x = 0$

$x(x^2 - 2x + 1) = 0 \quad \Rightarrow \quad x(x-1)(x-1) = 0$

$x = 0$ or $x = 1$

$\frac{d^2y}{dx^2} = 36x^2 - 48x + 12$

When $x = 0$, $\frac{d^2y}{dx^2} = 12 > 0$

$\therefore x = 0$ is a minimum

When $x = 1$, $\frac{d^2y}{dx^2} = 36 \times 1^2 - 48 \times 1 + 12 = 0$

Since $\frac{d^2y}{dx^2} = 0$ at $x = 1$ we must use the gradient change method to determine the nature of the stationary point

When $x = 0.9$, $\frac{dy}{dx} = 12 \times 0.9^3 - 24 \times 0.9^2 + 12 \times 0.9 > 0$

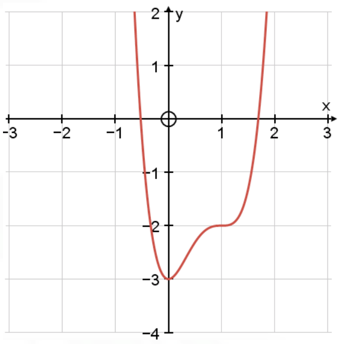
When $x = 1.1$, $\frac{dy}{dx} = 12 \times 1.1^3 - 24 \times 1.1^2 + 12 \times 1.1 > 0$

Since $\frac{dy}{dx}$ is the same sign either side of $x = 1$, the point is neither a maximum nor a minimum.

When $x = 0, y = -3$

When $x = 1, y = 3 - 8 + 6 - 3 = -2$

$(0, -3)$ is a minimum; $(1, -2)$ is neither maximum or minimum



N.B. The case of $\frac{d^2y}{dx^2} = 0$ will be covered in depth in the A2 course.

(c) $y = x^2 + \dfrac{16}{x^2} = x^2 + 16x^{-2}$

$\dfrac{dy}{dx} = 2x - 32x^{-3} = 2x - \dfrac{32}{x^3}$

A SP occurs when $\dfrac{dy}{dx} = 0$ so $2x - \dfrac{32}{x^3} = 0$

$2x = \dfrac{32}{x^3} \quad \Rightarrow \quad x^4 = 16 \quad \Rightarrow \quad x = \pm 2$

$\dfrac{d^2y}{dx^2} = 2 + 96x^{-4} = 2 + \dfrac{96}{x^4}$

When $x = 2$, $\dfrac{d^2y}{dx^2} = 2 + \dfrac{96}{2^4} > 0 \quad \therefore x = 2$ is a minimum

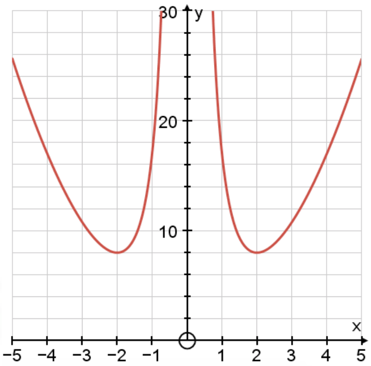
When $x = 2$, $\dfrac{d^2y}{dx^2} = 2 + \dfrac{96}{(-2)^4} > 0$

$\therefore x = -2$ is a minimum

When $x = 2$, $y = 2^2 + \dfrac{16}{2^2} = 8$

When $x = -2$, $y = (-2)^2 + \dfrac{16}{(-2)^2} = 8$

(−2, 8) and (2, 8) are both minima



E.g. 3 Show that the graph of the function $f(x) = x^5 + 3x + 2$ has no stationary points.

Working: $f'(x) = 5x^4 + 3$

A stationary point occurs when $f'(x) = 0$: $\quad 5x^4 + 3 = 0$

$5x^4 = -3 \quad \Rightarrow \quad x^4 = -\dfrac{3}{5}$

We cannot even root a negative number and get a real number so no real solutions.

Video: Nature of stationary points

Solutions to Starter and E.g.s

Exercise
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