Optimisation (triple)

Starter

1. (Review of last lesson) Consider the curve given by $y = x^4 + kx^3 + x^2 + 17$.
   (a) Find the range of values of k if the curve has only 1 stationary point.
   (b) Find the coordinates of the stationary point and say whether it's a maximum or minimum.

Working: (i) $\dfrac{dy}{dx} = 4x^3 + 3kx^2 + 2x$

A SP occurs when $\dfrac{dy}{dx} = 0$ so $4x^3 + 3kx^2 + 2x = 0$

$x(4x^2 + 3kx + 2) = 0$

For only 1 stationary point $4x^2 + 3kx + 2 \neq 0$

i.e. there are no roots so $b^2 - 4ac < 0$

$a = 4 \qquad b = 3k \qquad c = 2$

$b^2 - 4ac = (3k)^2 - 4 \times 4 \times 2 < 0$

$9k^2 - 32 < 0$

Roots are $k = \pm \dfrac{4\sqrt{2}}{3}$

Coefficient of k^2 is +ve so concave-up

$< 0 \Rightarrow$ below the $x-$axis

$-\dfrac{4\sqrt{2}}{3} < k < \dfrac{4\sqrt{2}}{3}$

(ii) The stationary point occurs when $x = 0$

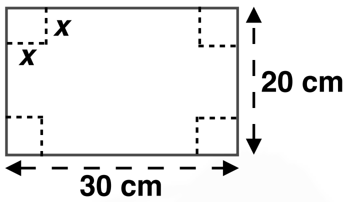
When $x = 0, y = 17$

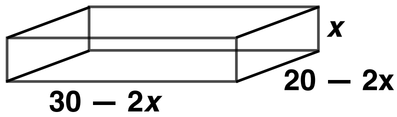
$\dfrac{d^2y}{dx^2} = 12x^2 + 6kx + 2$

When $x = 0$, $\dfrac{d^2y}{dx^2} = 2 > 0$ so (0, 17) is a minimum

E.g. 1 A open top box is made from a sheet of paper 30cm by 20cm by cutting out squares from the corners of the sheet of paper and folding the sides up. Find maximum volume of the box.

Working:





Let a square of side x be removed from each corner and let V be the volume of the open top box.

$V = x(30 - 2x)(20 - 2x)$
$\quad = 600x - 100x^2 + 4x^3$

$\frac{dV}{dx} = 600 - 200x + 12x^2$

A maximum occurs when $\frac{dV}{dx} = 0$ so $600 - 200x + 12x^2 = 0$

Solving the quadratic gives $x = 3.92$ or $x = 12.7$

$\frac{d^2V}{dx^2} = -200 + 24x$

When $x = 3.92$, $\frac{d^2V}{dx^2} < 0$ so $x = 3.92$ gives a maximum volume

When $x = 12.7$, $\frac{d^2V}{dx^2} > 0$ so $x = 12.7$ gives a minimum volume

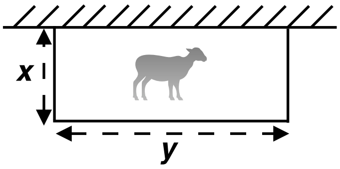
When $x = 3.92$, $V = 3.92(30 - 2 \times 3.92)(20 - 2 \times 3.92) \approx 1056$

So the maximum volume of the box is about 1056 cm³.

N.B. In practical problems, it is usually possible to discard one of the values immediately. In the example above, $x < 10$ otherwise the box would disappear so $x \neq 12.7$. It is fine to use such arguments to discard values but proof that the other value gives a maximum (or minimum) is still needed.

E.g. 2 A farmer has 80 m of fencing to mark out a rectangular enclosure, of dimensions x by y, against a brick wall. Find the maximum area that can be enclosed.

Working:



It doesn't matter which sides are x and y

Let the area of the enclose be, A so $A = xy$
Perimeter of enclose is 80 m so $2x + y = 80$
We cannot differentiate $A = xy$ because there are 2 variable on the RHS
From the perimeter equation $y = 80 - 2x$
Replace y by $80 - 2x$ in the formula for area: $A = x(80 - 2x)$
$A = 80x - 2x^2$
$\frac{dA}{dx} = 80 - 4x$
A maximum occurs when $\frac{dA}{dx} = 0$ so $80 - 4x = 0$
Solving gives $x = 20$
$\frac{d^2A}{dx^2} = -4 < 0$ so $x = 20$ is a maximum
When $x = 20, A = 20(80 - 2 \times 20 = 800$
So the maximum area is 800 m^2.

E.g. 3 A closed rectangular box is made of thin sheet metal and its length is three times its width.

(a) If the volume of the box is 288 cm³, show that its surface area is equal to $\left(\dfrac{768}{x} + 6x^2\right)$ cm², where x is the width of the box.

(b) Find by differentiation the dimensions of the box of least surface area.

Working: (a) Let the height of the box be h.

Since x is the width of the box, $3x$ is the length of the box.

The volume of the box is 288: $3x \times x \times h = 288$

$$x^2 h = 96$$

Let the surface area of the be S: $S = 2(3xh + xh + 3x^2)$

$$S = 8xh + 6x^2$$

We need to replace h (or x) is the equation.

Rearranging $x^2 h = 96$: $h = \dfrac{96}{x^2}$

Substitute into $S = 8xh + 6x^2$: $S = 8x \times \dfrac{96}{x^2} + 6x^2$

$$S = \frac{768}{x} + 6x^2$$

(b) $S = 768x^{-1} + 6x^2$

$$\frac{dS}{dx} = -768x^{-2} + 12x$$

A SP occurs when $\dfrac{dS}{dx} = 0$: $-768x^{-2} + 12x = 0$

$$12x = \frac{768}{x^2}$$

$$x^3 = 64$$

$$x = 4$$

$$\frac{d^2S}{dx^2} = 1536x^{-3} + 12 = \frac{1536}{x^3} + 12$$

When $x = 4$, $\dfrac{d^2S}{dx^2} > 0$ so $x = 4$ gives a minimum.

When $x = 4$: $3x = 12$ and $h = \dfrac{96}{4^2} = 6$

The dimensions of the closed box are 4 cm by 6 cm by 12 cm

Video: Applications of stationary points
Video: Maximising the volume of a box
Video: Minimising surface area of a box

Applications of stationary points EQ

Solutions to Starter and E.g.s

Exercise

p285 14C Qu 2, 4, 5 (S = …), 7, 10 (S = …)

Surface area of a cylinder $= 2\pi r^2 + 2\pi rh$