Proof by exhaustion

Starter

1. *(Review of last lesson)*
   (a) Disprove by counter-example that, for numbers greater than one, 1 less than a square number cannot be prime is false.
   (b) Ignoring the example given in (a), prove that 1 less than a square number cannot be prime.

Working: (a) 4 is a square number and one less that it is 3, which is a prime number.
Therefore, the statement that "1 less than a square number cannot be prime" is false

(b) A square number is of the form x^2 so one less than a square number is of the form $x^2 - 1$.
$x^2 - 1 = (x-1)(x+1)$ so $x^2 - 1$ has two factors
For $x^2 - 1$ to be prime, the smallest factor must be 1
i.e. $x - 1 = 1$ $\quad\therefore x = 2$ $\quad$ so $\quad x^2 = 4$
Since the example from (a) can be ignored, 1 less than a square number cannot be prime.

2. (a) Prove the statement that $n^2 - n - 1$ is a prime number for $3 \le n \le 7$, where n is an integer.
   (b) Disprove the statement that $n^2 - n - 1$ is always a prime number for $n > 2$, where n is an integer.

Working: (a)
$n = 3$: $\quad n^2 - n - 1 = 3^2 - 3 - 1 = 5$ $\quad$ prime
$n = 4$: $\quad n^2 - n - 1 = 4^2 - 4 - 1 = 11$ $\quad$ prime
$n = 5$: $\quad n^2 - n - 1 = 5^2 - 5 - 1 = 19$ $\quad$ prime
$n = 6$: $\quad n^2 - n - 1 = 6^2 - 6 - 1 = 29$ $\quad$ prime
$n = 7$: $\quad n^2 - n - 1 = 7^2 - 7 - 1 = 41$ $\quad$ prime

(b) $n = 8$: $\quad n^2 - n - 1 = 8^2 - 8 - 1 = 55$ $\quad$ not prime
So $n^2 - n - 1$ is not always a prime number for $n > 2$.

E.g. 1 Prove that for every integer n such that $0 \le n < 4$, $2^{n+2} > 3^n$.

Working:
$n = 0$: $\quad 2^2 = 4 > 1 = 3^0$
$n = 1$: $\quad 2^{1+2} = 8 > 3 = 3^1$
$n = 2$: $\quad 2^{2+2} = 16 > 9 = 3^2$
$n = 3$: $\quad 2^{3+2} = 32 > 27 = 3^3$
Hence, $2^{n+2} > 3^n$ for $0 \le n < 4$.

E.g. 2 Prove that the product of any three consecutive integers is even.

Working: Let the consecutive integers be even, odd, even e.g. $2n$, $2n+1$, $2n+2$
Product $= 2n \times (2n+1) \times (2n+2) = 2\left[n(2n+1)(2n+2)\right] = 2p$
which is the form of an even number.
Let the consecutive integers be odd, even, odd e.g. $2n-1$, $2n$, $2n+1$
Product $= (2n-1) \times 2n \times (2n+1) = 2\left[n(2n-1)(2n+1)\right] = 2q$
which is the form of an even number.
Therefore, the product of any three consecutive integers is even.

E.g. 3 Prove that, for any integer x, the value of $x^3 + x + 1$ is always an odd integer.

Working: Let x be even i.e. $x = 2p$
Then $x^3 + x + 1 = (2p)^3 + 2p + 1 = 8p^3 + 2p + 1 = 2(4p^3 + p) + 1$
which is of the form $2m + 1$ i.e. an odd number.
Let x be odd i.e. $x = 2q + 1$
Then $x^3 + x + 1 = (2q+1)^3 + 2q + 1 + 1$
$= 8q^3 + 12q^2 + 6q + 1 + 2q + 1 + 1$
$= 8q^3 + 12q^2 + 8q + 2 + 1$
$= 2(4q^3 + 6q^2 + 4q + 1) + 1$
which is of the form $2n + 1$ i.e. an odd number.
Hence the value of $x^3 + x + 1$ is always an odd integer.

Video: **Proof by exhaustion and deduction**

Solutions to Starter and E.g.s

Exercise
p13 1E Qu 1-8, (9-10 red)