Using identities to solve equations

Starter

1. *(Review of last lesson)* Solve the equation $4\cos^2\theta - 7\cos\theta = 2$ for $0° \le \theta \le 360°$.

Working: $4\cos^2\theta - 7\cos\theta = 2 \quad \Rightarrow \quad 4\cos^2\theta - 7\cos\theta - 2 = 0$

$(4\cos\theta + 1)(\cos\theta - 2) = 0$

$\cos\theta = -\dfrac{1}{4}$ or $\cos\theta = 2$

$\cos\theta = 2 \quad \Rightarrow \quad$ No solution since $-1 \le \cos\theta \le 1$

$\cos\theta = -\dfrac{1}{4}$

$\cos^{-1}\dfrac{1}{4} \approx 75.52°$

$-\dfrac{1}{4}$ is $-$ve

cos is $-$ve in the **S** & **T** quadrants

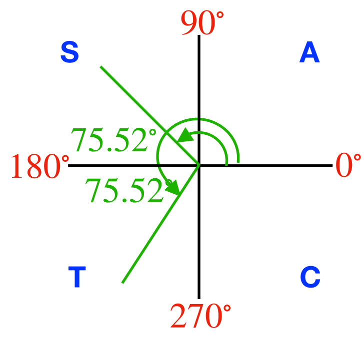
Draw the angle from the horizontal

Measure anti-clockwise from the $+$ve $x-$axis

$\theta = 180° - 75.52°$ or $180° + 75.52°$

$\theta = 104°$ or $256°$ (3 s.f.)

The required angles are $104°, 256°$.



2. Solve $3\cos\theta\sin\theta = 5\cos\theta$ for $-180° \le \theta \le 180°$.

Working: $3\cos\theta\sin\theta = 5\cos\theta \quad \Rightarrow \quad \cos\theta(3\sin\theta - 5) = 0$

$\cos\theta = 0$ or $\sin\theta = \dfrac{5}{3}$

$\sin\theta = \dfrac{5}{3} \quad \Rightarrow \quad$ No solution since $-1 \le \sin\theta \le 1$

$\cos\theta = 0 \quad \Rightarrow \quad \theta = \pm 90°$

3. Using a suitable trigonometric identity, solve $4\sin\theta = \cos\theta$ for $0° \le \theta \le 360°$.

Working: $4\sin\theta = \cos\theta \quad \Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{4} \quad \Rightarrow \quad \tan\theta = \dfrac{1}{4}$

$\tan^{-1}\dfrac{1}{4} \approx 14.0°$

$\dfrac{1}{4}$ is $+$ve

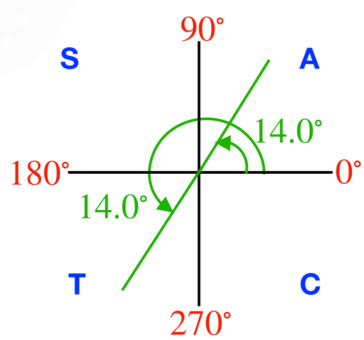
tan is $+$ve in the **A** & **T** quadrants

Draw the angle from the horizontal

Measure anti-clockwise from the $+$ve $x-$axis

$\theta = 14.0°$ or $180° + 14.0°$

$\theta = 14.0°$ or $194°$ (3 s.f.)



4. Using a suitable trigonometric identity, solve $2\sin^2\theta + 3\cos\theta - 3 = 0$ for $0° \le \theta \le 360°$.

Working: Using $\sin^2\theta = 1 - \cos^2\theta$:

$$2(1-\cos^2\theta) + 3\cos\theta - 3 = 0$$
$$2\cos^2\theta - 3\cos\theta + 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta - 1) = 0$$
$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = 1$$

From the graph, $\cos\theta = 1 \;\Rightarrow\; \theta = 0°, 360°$

$\cos\theta = \frac{1}{2}$

$\cos^{-1}\frac{1}{2} = 60°$

$\frac{1}{2}$ is +ve

cos is +ve in the **A** & **C** quadrants

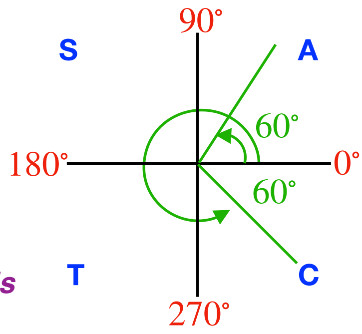
Draw the angle from the horizontal

Measure anti-clockwise from the +ve $x-$axis

$\theta = 60°$ or $360° - 60°$

$\theta = 60°$ or $300°$

The required angles are $0°, 60°, 300°, 360°$.



E.g. 1 Solve these equations for $-180° \le \theta \le 180°$ giving your answer to a suitable accuracy:

(a) $8\cos\theta = -3\sin\theta$ (b) $5\sin\theta - 7\cos\theta = 0$

Working: (a) $8\cos\theta = -3\sin\theta \;\Rightarrow\; \dfrac{\sin\theta}{\cos\theta} = -\dfrac{8}{3} \;\Rightarrow\; \tan\theta = -\dfrac{8}{3}$

Ignore the −ve sign

$\tan^{-1}\frac{8}{3} = 69.4°$

$-\frac{8}{3}$ is −ve

tan is −ve in the **S** & **C** quadrants

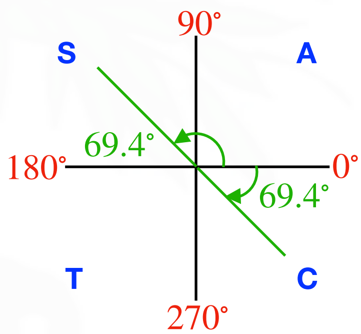
Draw the angle from the horizontal

Measure anti-clockwise for S quadrant

Measure clockwise for C quadrant

$\theta = 180° - 69.4°$ or $-69.4°$

$\theta = 111°$ or $-69.4°$ (3 s.f.)



(b) $5\sin\theta - 7\cos\theta = 0 \;\Rightarrow\; \dfrac{\sin\theta}{\cos\theta} = \dfrac{7}{5} \;\Rightarrow\; \tan\theta = \dfrac{7}{5} = 1.4$

$\tan^{-1}1.4 = 54.46°$

tan is +ve in the **A** & **T** quadrants

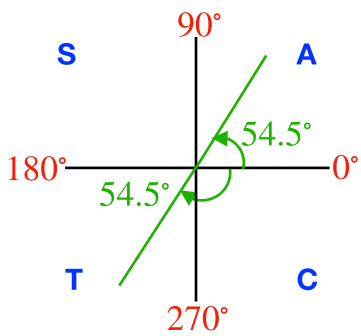
Draw the angle from the horizontal

Measure anti-clockwise for A quadrant

Measure clockwise for T quadrant

$\theta = 54.46°$ or $-180° + 54.46°$

$\theta = 54.46°$ or $-126°$



Solving trigonometric equations requiring $\sin^2\theta + \cos^2\theta \equiv 1$

When a quadratic equation includes a quadratic term in $\sin$ or $\cos$ and a simple term in $\cos$ or $\sin$, replace the quadratic term using the identities:

$$\cos^2\theta \equiv 1 - \sin^2\theta$$
$$\textbf{or } \sin^2\theta \equiv 1 - \cos^2\theta$$

Use $\cos^2\theta \equiv 1 - \sin^2\theta$ or $\sin^2\theta \equiv 1 - \cos^2\theta$:

When: a ***quadratic*** equation includes a term in $\sin^2$ or $\cos^2$ and a simple multiple of $\cos$ or $\sin$.

How: replace the quadratic term.

E.g. 2 Solve these equations for $-180^\circ \leq \theta \leq 180^\circ$ giving your answer to a suitable accuracy:

(a) $6\cos^2\theta + \sin\theta - 5 = 0$ (b) $3\cos\theta + 7 = 9\sin^2\theta$

Working: (a) $6\cos^2\theta + \sin\theta - 5 = 0 \Rightarrow$

$$6(1 - \sin^2\theta) + \sin\theta - 5 = 0$$
$$6\sin^2\theta - \sin\theta - 1 = 0$$
$$(3\sin\theta + 1)(2\sin\theta - 1) = 0$$
$$\sin\theta = \frac{1}{2} \text{ or } \sin\theta = -\frac{1}{3}$$

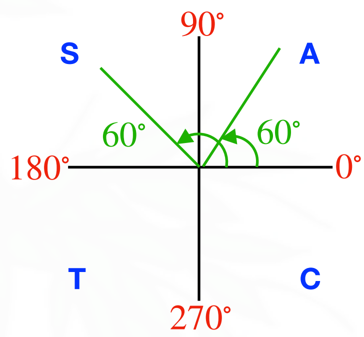
$\sin\theta = \dfrac{1}{2}$

$\sin^{-1}\dfrac{1}{2} \approx 30^\circ$

$\dfrac{1}{2}$ is +ve

sin is +ve in the **A** & **S** quadrants

Draw the angle from the horizontal
Measure anti-clockwise from the +ve $x-$axis



$\theta = 30^\circ$ or $180^\circ - 30^\circ$

$\theta = 30^\circ$ or 150°

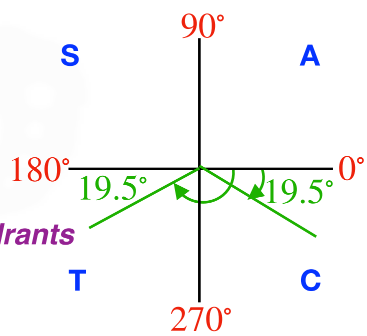
$\sin\theta = -\dfrac{1}{3}$

Ignore the negative sign

$\sin^{-1}\dfrac{1}{3} \approx 19.47^\circ$

$-\dfrac{1}{3}$ is −ve $\Rightarrow$ **T** & **C** quadrants

Draw the angle from the horizontal
Measure anti-clockwise for S & T quadrants



$\theta = -19.47^\circ$ or $-180^\circ + 19.47^\circ$

$\theta = -19.5^\circ$ or -161°

The required angles are

$-161^\circ, -19.5^\circ, 30^\circ$ or 150°

(b) $3\cos\theta + 7 = 9\sin^2\theta \quad \Rightarrow \quad 3\cos\theta + 7 = 9(1 - \cos^2\theta)$

$$9\cos^2\theta + 3\cos\theta - 2 = 0$$
$$(3\cos\theta + 2)(3\cos\theta - 1) = 0$$
$$\cos\theta = -\frac{2}{3} \text{ or } \cos\theta = \frac{1}{3}$$

$\cos\theta = \frac{1}{3}$

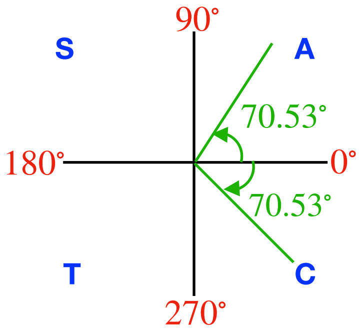
$\cos^{-1}\frac{1}{3} \approx 70.53^\circ$

$\frac{1}{3}$ is $+$ve

cos is $+$ve in the **A** & **C** quadrants

Draw the angle from the horizontal
Measure anti-clockwise for A quadrant
Measure clockwise for C quadrant

$\theta = -70.5^\circ$ or 70.5° (3 s.f.)



$\cos\theta = -\frac{2}{3}$

Ignore the negative sign

$\cos^{-1}\frac{2}{3} \approx 48.2^\circ$

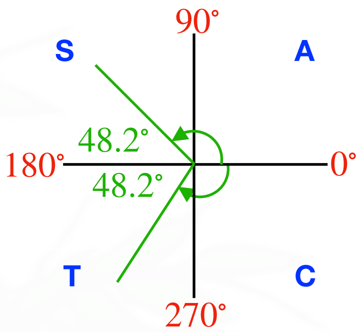
$-\frac{2}{3}$ is $-$ve

cos is $-$ve in the **S** & **T** quadrants

Draw the angle from the horizontal
Measure anti-clockwise for S quadrant
Measure clockwise for T quadrant

$\theta = 180^\circ - 48.2^\circ$ or $-180^\circ + 48.2^\circ$

$\theta = 132^\circ$ or -132°



The required angles are $-132^\circ, -70.5^\circ, 70.5^\circ$ or 132°

Video: Solving equations using identities

Exam questions: Trigonometric equations

Solutions to Starter and E.g.s

Exercise
p199 10H Qu 1i, 2i, 3-12