The Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) \times e^{-i2\pi ft} \, dt$$

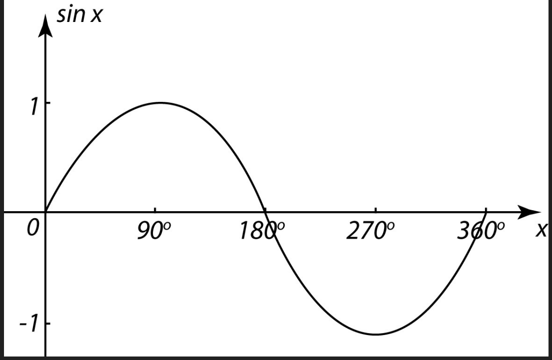
- The Fourier transform is a method of decomposing complex waves into the simple sine waves that it is formed form.
- In this presentation I am going to talk you through the derivation of this equation and its uses.

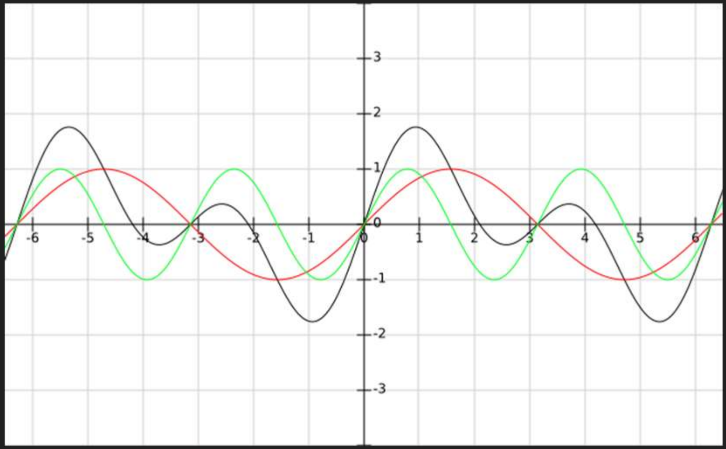
Joseph Fourier

- The Fourier transform is an equation created by Joseph Fourier (March 21st 1768 – May 16th 1830).
- Fourier was a French Mathematician who made drastic advances in mathematical physics. He sought to understand heat and to find the mathematical laws that it follows. He predicted the theory of heat would form one of the most important branches of general physics.
- And he was exactly right because his mathematical understanding of how heat varies has allowed us to understand everything that can be described as a wave such as heat, sound and light.

Sine Waves

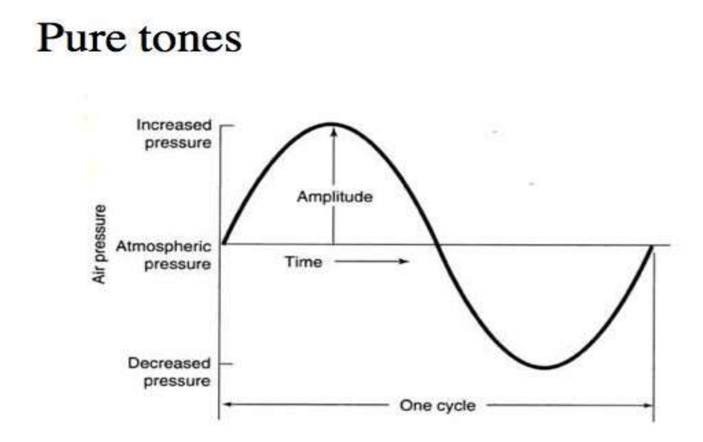
- The simplest wave is the sin wave.
- But when you combine more than one sine waves of different frequencies or that are out of phase (At different points along their repeating cycle) they form much more complicated waves. This is because at any value of t the overall pressure is equal to the sum of the pressures of the individual sine waves at that point t.
- This is shown to the right where the two different sine waves (red and green) combine to form the complex Blue wave.
- Using the Fourier transform we can find the equations of the red and green waves from the blue wave.

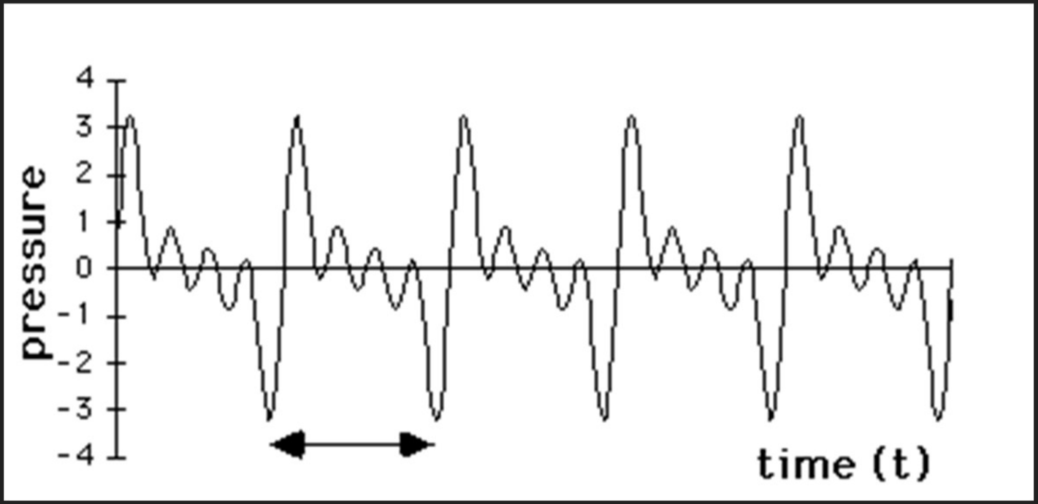




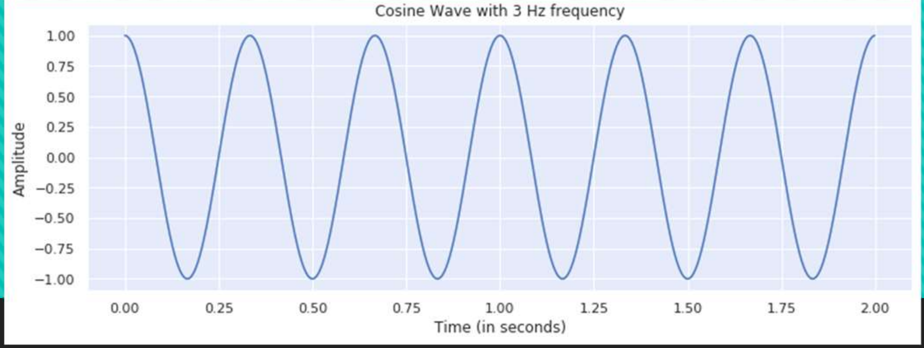
Sound Waves

- For example a tuning fork will produce a sound wave such as the function on the left. This is because the vibrations of the fork are a simple and constant back and forth motion. However, the sound waves of harmonics produced by musical instruments and your voice are much more complicated such as the function on the right.

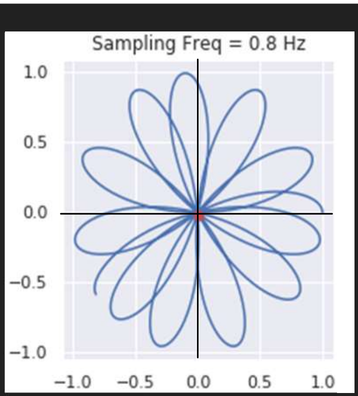


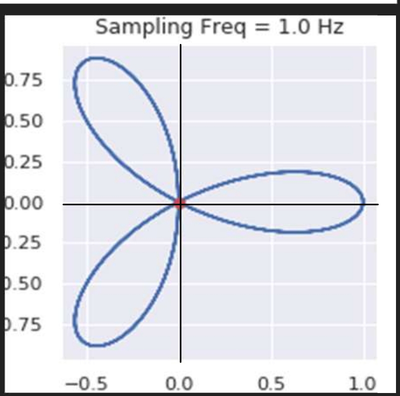


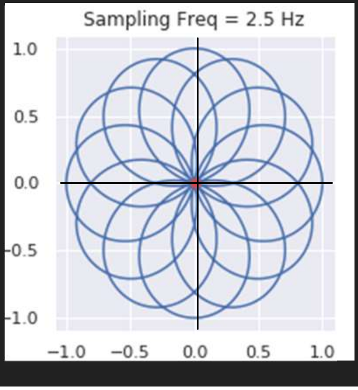
Deriving The Transform

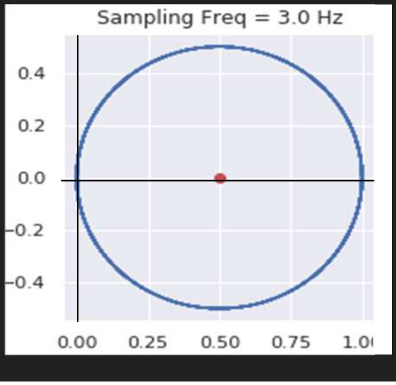


- The equation may seem daunting at first but we’re going to break it down step by step.
- First lets take a simple sine wave of frequency 3Hz from t=0 to t=2.
- We can then plot this as a new function rotating round a point where the amplitude of the original wave corresponds to the distance from that point and the frequency of the new function is how many revolutions round the point occur each second.
- Varying the frequency of this new function can make some complicated shapes but notice at f=1 and f=3 the shapes become simple. This is because at f=1 the frequencies of the new and original function are such that there are exactly 3 full wave cycles (out from the point and back again) in one second and so one revolution round the point. At f=3 each rotation is exactly one wave cycle as there are 3 rotations per second and 3Hz mans there are 3 wave cycles per second.
- The red dot shows the average point of all the points on the new function. Notice when the frequencies of the two functions are equal the average point moves a considerable amount from (0,0), the centre point of the function. This is important as if we can find the frequencies at which this occurs we will be able to find the frequencies of the sine waves a complex wave is made from.



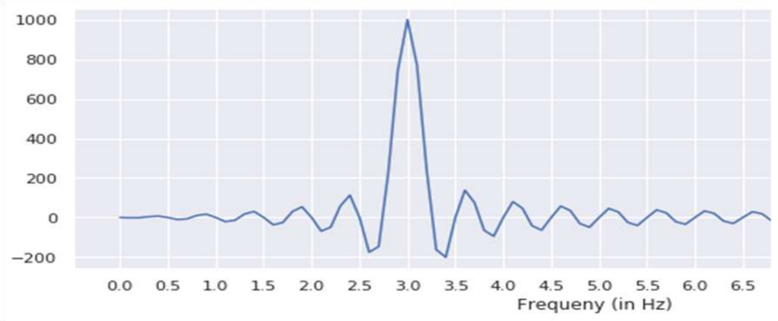


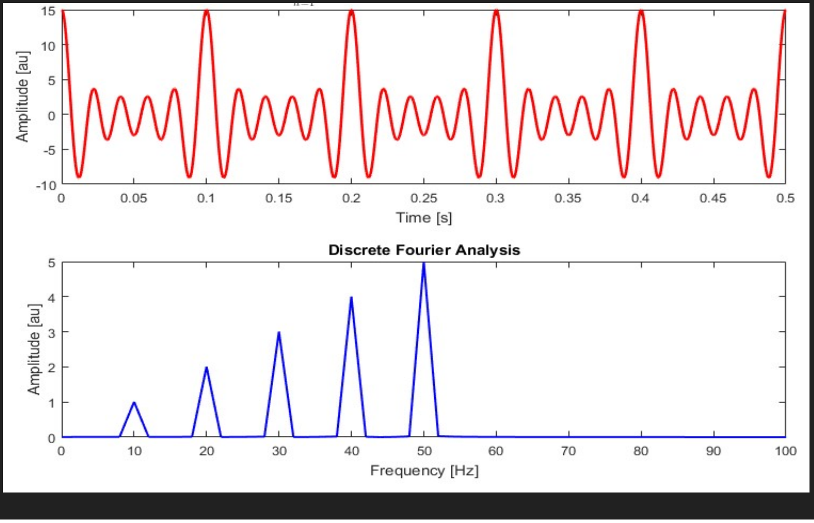




The Actual Transform Itself

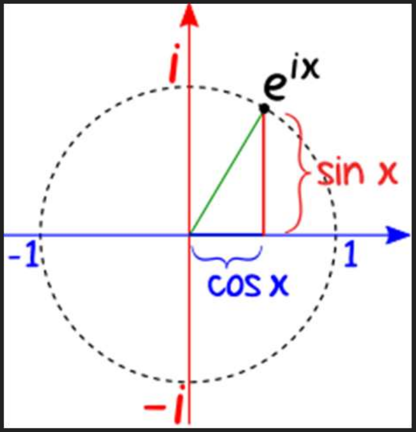
- So since this property of the average point occurs at the frequencies of the sine waves a complex wave is made of if we plot the position of the average point against the specific frequency of the circular function you'll get a graph similar to the one on the right. This is the Fourier transform. Yes, eventually we've got there. The equation you saw at the beginning converts a normal wave into this so we can observe the frequencies present. So this is what we are trying find and is denoted by x(f).
- You can see that the clear peak at 3Hz shows us that the frequency of the simple sine wave is 3Hz which we know to be true as that was our start product.
- For complex waves made of several frequencies of sine waves the Fourier transform will have several peaks. Note that the figure in the bottom right shows both the original complex wave and its Fourier transform and that the transform has been flattened out around the peaks for a clearer observation.





Equation of the circular function

- But how do we find the average point?
- Well to do that we must find an equation to describe this new circular function. If we use the complex plane to map out the function we can use Euler's formula to find the coordinates of the points along the function.
- Euler's formula states **$e^{ix} = \cos(x) + i\sin(x)$**. So for a circle of radius 1 and centre 0+0i **e^{ix}** is the point on that circle after moving x radians anticlockwise round that circle. This is shown in the diagram to the right.
- So to describe one full rotation we can use **$e^{2\pi i}$** since 2π radians = 360˚. To describe rotation at a rate of one cycle per second we can use **$e^{2\pi i t}$** where t=time passed. And by using **$e^{2\pi i f t}$**, where f = the frequency of the new function, we can describe rotation that occurs at a time period equal to the frequency. For example when f=3 we will get one full revolution every 3 seconds since ft = 1/3 X 3 = 1. To describe clockwise motion we simply use **$e^{-2\pi i f t}$**.
- Finally if the simple sine wave from the previous slide is denoted by x(t) then in our new function the distance from the centre point will be equal to x(t) since all the points described by our equation are at a distance 1 from the centre and the amplitude of the original wave is equal to x(t).
- Therefore, the equation that plots all points in the new circular function at a given frequency after t time has past is **$x(t)e^{-2\pi i f t}$**.



Finally!

- Now that we have an equation that describes the circular function we can find the average point of that function at any given frequency after t time has past.
- This can be simply done by taking an integral of the function then dividing by the time period that integral is taken with. This will give us the exact position of the average point. However, we are only interested in the variation of the distance of the average point form the centre point and when it becomes unusually distant. In this case we can avoid dividing by the time period to make the equation simpler. When we consider this expression for all finite time intervals the integral becomes such that the upper bound is infinity and lower bound is –infinity.
- And with that we get the Fourier transform as: $X(f) = \int_{-\infty}^{\infty} x(t) \times e^{-i2\pi f t}\, dt$

Uses Of The Transform + The Inverse Transform

- This process can be simply reversed by applying the inverse Fourier Transform shown below. Applying this to the transform will give you the original wave back.
- This has many uses, one of which is in sound editing. Say your sound has a very annoying high pitch to it. You can apply the transform, locate the high frequency causing that sound, remove the peak at that point on the transform and then apply the inverse transform to return to the original sound wave but without the annoying high pitch.
- Another use is in Infra Red Spectroscopy. This is a method of analysing substances to see what molecular bonds are present. You measure the amount of IR radiation absorbed by the substance and the Fourier transform allows us to analyse the intensity of several wavelengths of IR radiation detected that has passed through the substance as apposed to single wavelengths. This makes it much quicker and more efficient.

$$x(t) = \frac{\int_{-\infty}^{\infty} X(f) \times e^{i2\pi f t}\, dt}{2\pi}$$