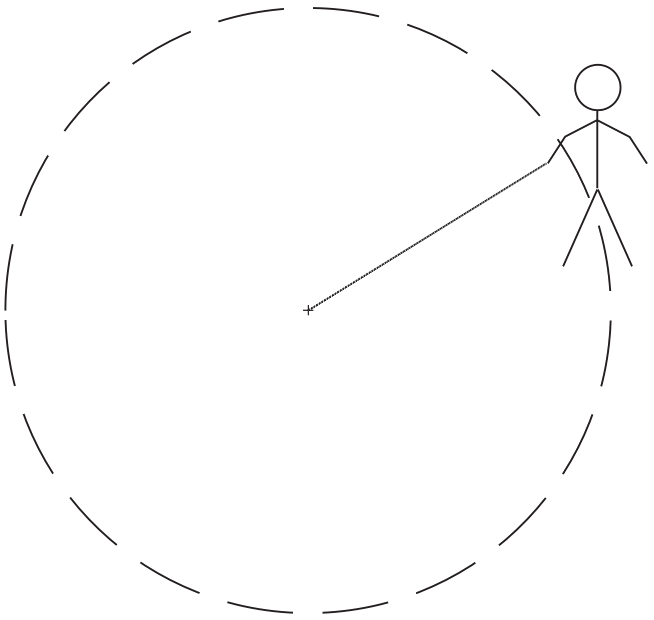
UNIT 16 *Circles and Cylinders* Activities

Activities

ACTIVITY 16.1 *Estimating π (Practical Method)*

One end of a rope or length of string is attached to a fixed point A pupil holds the other end and paces out a circle.



The length of the string is also measured in paces by the pupil.

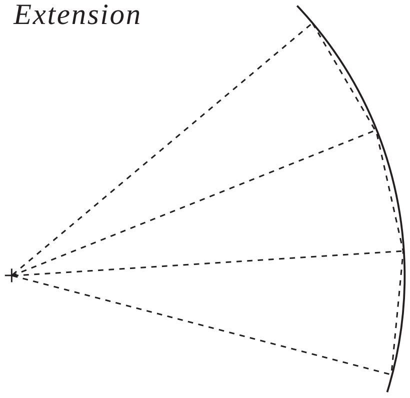
1. Collect data for the radius and circumference, in paces, for several different pupils' circles.

2. For each set of data, calculate the quantity

$$\frac{\text{circumference}}{2 \times \text{radius}}$$

and find its average value.

Extension



Estimate the area of the circle by using sectors for each pace, and for each set of data, calculate

$$\frac{\text{area}}{(\text{radius})^2}$$

This will give an estimate of π, which should improve in accuracy for small paces.

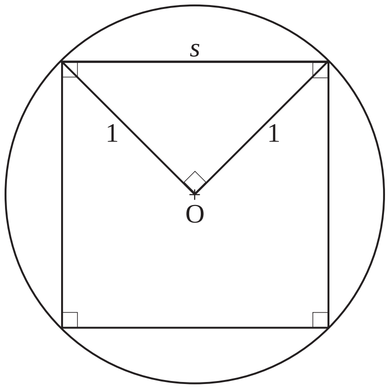
ACTIVITY 16.2 *Estimating π (Theoretical Method)*

The method outlined here was that used by *Archimedes of Syracuse* (287-212 BC) to obtain approximations to π, to whatever accuracy is needed.

We first note that for a circle of radius r, the ratio

$$\frac{\text{circumference}}{\text{diameter}} = \frac{2\pi r}{2r} = \pi$$

1st Approximation

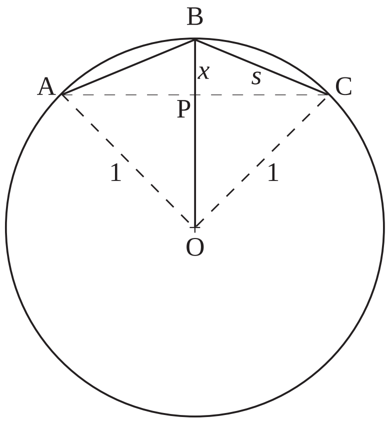


Consider a circle of unit radius with a square inscribed.

1. What is the length of the side, s, of the square?

2. As an approximation to the circumference, use $4 \times s$ and calculate the approximation for π using the formula above (it's not a very accurate approximation!)

2nd Approximation



Now we inscribe an 8-sided (regular) polygon (i.e. an octagon) inside the circle. ABC is part of the octagon. The line AC crosses the radius OB at P. Let $BP = x$.

3. What is the length of OP ?

4. Using Pythagoras' Theorem on triangle APO, show that
$$x = 1 - \frac{1}{\sqrt{2}}$$

5. Using Pythagoras' Theorem on triangle APB, show that
$$s\ (= AB) = \sqrt{2 - \sqrt{2}}$$

6. Now use $8 \times s$ for the circumference and calculate the next approximation for π using the formula above.

Extension

1. Using the same method, show that for a regular 16-sided polygon, the side length $= \sqrt{2 - \sqrt{2 + \sqrt{2}}}$. Hence determine the next approximation for π.

2. Generalise the results to show that, for example, for a 2048-gon,

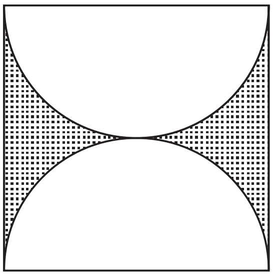
$$\pi \approx 1024\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}}}$$

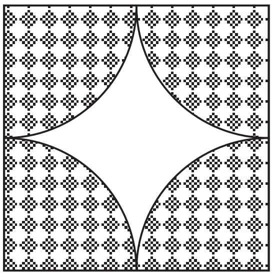
What value does this give for π ?

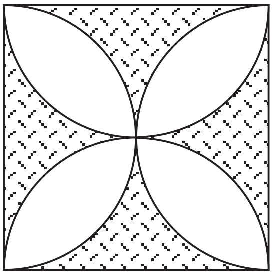
ACTIVITY 16.3 *Window Design*

When designing and costing stained glass windows, it is important to be able to calculate the *ratio* of colourless to coloured glass.

Examples of different designs are shown below. The shaded portions are to be coloured glass and the white portions are to be clear glass.

(a)


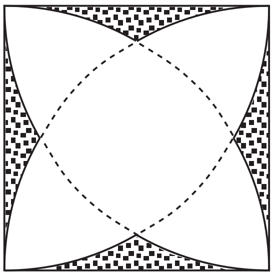
(b)


(c)


1. For each of the designs above, calculate the ratio of colourless to coloured area. Assume each is based on a square of side 20 cm and circles of *diameter* 20 cm

2. Design your own stained glass window and calculate the ratio of colourless to coloured area.

Extension



The design opposite is rather more complicated. The square has sides of length 20 cm but the arcs are parts of circles of radius 20 cm.

Calculate the ratio of colourless to coloured glass for this design.

(*Hint*: you might need to use Pythagoras' Theorem to solve this problem.)

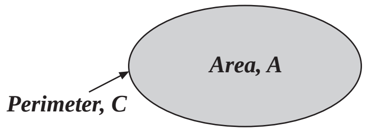
ACTIVITY 16.4 *Isoperimetric Quotient Numbers*

According to legend, Princess Dido, a Greek princess who was fleeing from the tyranny of her brother, landed at Constantinople in Turkey and grudgingly gained a concession from the natives that she could have

"*all the land that could be encompassed by the skin of an ox.*"

The natives expected her to find the biggest ox skin possible and claim land of the equivalent area, but she was far cleverer than that. She not only found the biggest ox skin, but cut it into very long, thin strips, joined all these together, placed them on the land, and claimed the whole area inside for herself. Her final problem was to decide in what shape she should put the strip down so as to enclose the largest possible area.

What is the answer to this problem?



A circle has the unique property of being symmetrical about any diameter. For a given perimeter length, C, the circle gives a maximum area enclosed - this rather obvious result is in fact exceedingly difficult to prove!

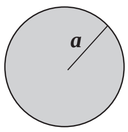
For a circle of radius a, $A = \pi a^2$ $C = 2\pi a$

We can find the relationship between A and C by eliminating a, giving

$$A = \pi \left(\frac{C}{2\pi} \right)^2 = \frac{C^2}{4\pi}$$

Hence for a circle,

$$\frac{4\pi A}{C^2} = 1$$



We define the *isoperimetric quotient* for any plane shape as

$$\text{I. Q.} = \frac{4\pi A}{C^2}$$

We shall see that its value is a measure of the shape's closeness to a circle. The I.Q. for a circle is 1, as we have shown above.

1. Calculate the I.Q. numbers for the following plane shapes:

   (a) square, side a,
   (b) rectangle of sides a, $2a$,
   (c) rectangle of sides a, $5a$,
   (d) equilateral triangle,
   (e) '3, 4, 5' triangle,
   (f) semicircle,
   (g) '5, 12, 13' triangle
   (h) regular pentagon.

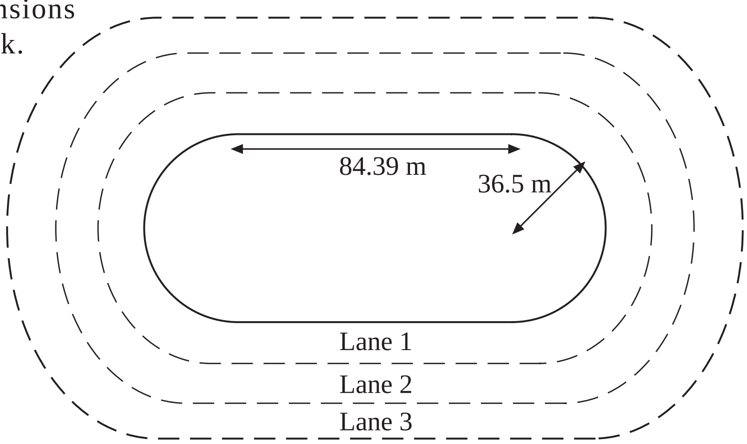
2. Draw up a table of I.Q. numbers in ascending order. What can you conjecture about I.Q. numbers?

ACTIVITY 16.5 *Track Layout*

The sketch shows the two main dimensions of a standard 400 metres running track.

1. Calculate the inside perimeter of this shape.

   Why do you think that it is *not* equal to 400 metres?



The inside runner cannot run at the very edge of the lane (there is normally an inside kerb) but let us assume that the athlete runs at a constant distance of, say, x cm from the inside edge.

2. What is the radius of the two circular parts run by the athlete in the inside lane?

3. Show that the total distance travelled, in centimetres, is

$$2\pi(3650 + x) + 16878$$

   and equate this to 40 000 cm to find a value for x. Is it realistic?

For 200 m and 400 m races, the runners run in specified lanes. Clearly, the further out you are the further you have to run, unless the starting positions are *staggered*.

The width of each lane is 1.22 m, and it is assumed that all runners (except the inside one) run about 20 cm from the inside of their lanes.

4. With these assumptions, what distance does the athlete in *Lane 2* cover when running one complete lap? Hence deduce the required stagger for a 400 m race.

5. What should be the stagger for someone running in *Lane 3* ?

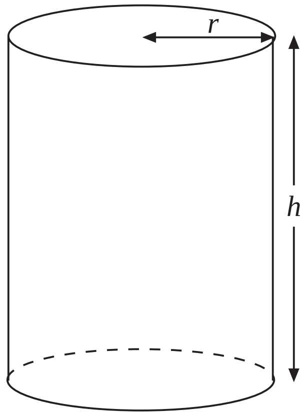
6. If there are 8 runners in the 400 m race, what is the stagger of the athlete in *Lane 8* compared with that in *Lane 1* ? Is there any advantage in being in *Lane 1*?

Extensions

1. The area available for a school running track is $90 \text{ m} \times 173 \text{ m}$. How many lanes could it have?

2. Design a smaller running track, with lanes, to fit an area $40 \text{ m} \times 90 \text{ m}$.

ACTIVITY 16.6 *Optimum Design for a Cylinder*



In practical situations, manufacturers of products sold in cans need to consider how best to design their container so that it holds a fixed volume.

For simplicity, we will assume that we need to design a can that will contain 1 litre ($= 1000$ cc).

1. If the can has radius r and height h, determine the relationship between r and h if the volume is 1 litre (i.e. 1000 cc).

   Hence show that, when h and r are measured in cm, $h = \dfrac{1000}{\pi r^2}$.

A possible criterion for the manufacturer is to minimise the amount of tin used to make the can.

2. Find an expression for the total surface area, s, of the can.

3. Determine the value of h, in cm to 1 d.p., when $r = 1, 2, \ldots, 10$ cm.

4. For each pair of values of r and h, determine the value of s, the corresponding total surface area, to the nearest cm^2.

5. Plot a graph of s against r, using the data from Question 4. Sketch a smooth curve through the points, and estimate the value of r that minimises s.

6. For your estimated minimum value of r found in Question 5, calculate the corresponding value of h. What do you conjecture?

Extension

1. Repeat the question for a different volume.

2. Are the results in line with what happens in practice? If not, explain why you think there is a mismatch between theory and practice.

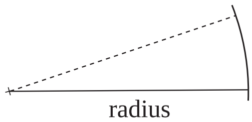
ACTIVITIES 16.1 - 16.3 *Notes for Solutions*

Notes and solutions given only where appropriate.

16.1 2. This should give an estimate of π. Note that, although different units are used (as the length of a pace will differ from pupil to pupil), this quantitly is non-dimensional.

Extension

You can estimate the area of each sector from



$$\frac{1}{2} \times \text{radius (in paces)} \times 1$$

and the total area is this quantity times the number of paces for the circumference.

16.2 1. $s = \sqrt{2}$

2. $\pi \approx 2.8284\ldots$

3. $\text{O P} = 1 - x$

6. $\pi \approx 3.0614\ldots$

Extension

1. $\pi \approx 3.1214\ldots$

2. $\pi \approx 3.141592949\ldots$ (correct to first 6 decimal places !)

16.3 1. (a) colourless area $= \pi \times 10^2 = 100\pi$

coloured area $= 400 - 100\pi$

$$\text{ratio} = \frac{100\pi}{400 - 100\pi} = \frac{\pi}{4 - \pi} \quad (\approx 3.66)$$

(b) as (a), but the other way round,

$$\text{i.e. ratio} = \frac{4 - \pi}{\pi} \quad (\approx 0.273)$$

(c) colourless area $= 2 \times (400 - 100\pi)$

coloured area $= 400 - (800 - 200\pi)$

$= 200\pi - 400$

$$\text{ratio} = \frac{800 - 200\pi}{200\pi - 400} \quad (\approx 0.752)$$

Extension Not an easy task, and only for the most able (pupils and teachers)!
(ratio ≈ 4.762, but please let us know if you have an easy way of solving this; e-mail us to get our complex solution!)

ACTIVITIES 16.4 - 16.5 *Notes for Solutions*

16.4 1.

(a) $\frac{\pi}{4} \approx 0.785$ (b) $\frac{2\pi}{9} \approx 0.698$

(c) $\frac{5\pi}{36} \approx 0.436$ (d) $\frac{\pi}{3\sqrt{3}} \approx 0.605$

(e) $\frac{\pi}{6} \approx 0.524$ (f) $\frac{2\pi^2}{(\pi + 2)^2} \approx 0.747$

(g) $\frac{2\pi}{15} \approx 0.419$ (h) $\frac{\pi \tan 54°}{5} \approx 0.865$

2. One possible conjecture is that I.Q. $\to 1$ as the shape gets closer to a circle.

16.5 1. 398.11 m ; the athlete cannot run on the edge.

2. $(3650 + x)$ cm

3. About 30 cm; yes

4. 407.0385 m $\Rightarrow$ stagger of 7.04 m.

5. 414.7039 m $\Rightarrow$ further stagger of 7.66 m

6. 53.05 m. Advantage of Lane 1 is that you can see your opponents.

Extensions

1. 6 lanes

2. A 200 m track with only two lanes.

ACTIVITIES 16.6 *Notes for Solutions*

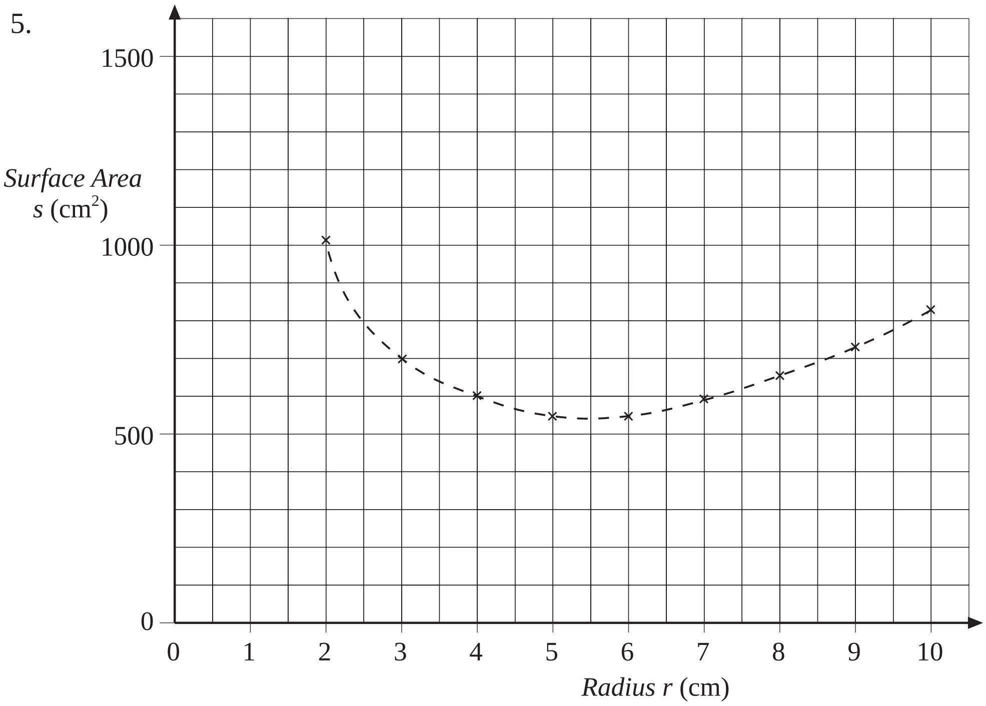
16.6 1. $1000 = \pi r^2 h$

2. $s = 2\pi r h + 2\pi r^2$

3. and 4.

r	1	2	3	4	5	6	7	8	9	10
h	318.3	79.6	35.4	19.9	12.7	8.8	6.5	5.0	3.9	3.2
s	2001	1025	723	601	557	560	594	652	731	828

5.



Minimum at (approx) $r = 5.5$ cm

6. Corresponding value of $h = 10.5$ cm

For optimum dimension $h \approx 2r$