

UNIT 2 *Factors*

Activities

Activities

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ACTIVITY 2.1

Sieve of Eratosthenes

The *Sieve of Eratosthenes* can be used to find prime numbers. In this activity, we will use the method to find all the prime numbers that are less than 100.

1. You will need a grid like this one. First shade the number 1 because it is *not* a prime number.

2. The next number is 2; this is the first prime number. Now shade all the other multiples of 2. This process has been started on the grid opposite.

3. The next number is 3. This is the second prime number. Now shade all the other multiples of 3. This process has been started on the grid below.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

4. The next prime is 5.

5. Keep repeating this process of shading the multiples of all prime numbers less than 10. Eventually only the prime numbers less than 100 will remain unshaded on your grid.

Extension

1. Why do you consider only multiples of primes *less than 10* ?
2. If you wanted to find all prime numbers less than 400, the multiples of which primes would need to be shaded? What size grid would you need?

ACTIVITY 2.2

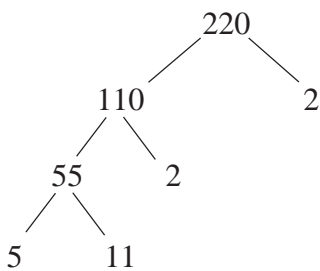
Perfect Numbers

A number is called a *perfect number* if all its factors (excluding the number itself) add up to the number.

1. Write down the factors of 6 and show that 6 is a perfect number.
2. Check to see if any of the numbers 2 to 5 are perfect.
3. Explain why no prime numbers are perfect numbers.
4. Copy and complete this table to find the next perfect number after 6.

Number	Factors	Sum of Factors (excluding number)
7	1, 7	$1 = 1$
8	1, 2, 4, 8	$1 + 2 + 4 = 7$
9	1, 3, 9	$1 + 3 = 4$
10		
...		
...		
30		

5. A factor tree can be used to find the factors of a number. For example, consider 220.



Factors of 220

- 1
- 2
- 5
- 11
- $2 \times 2 = 4$
- $2 \times 5 = 10$
- $2 \times 2 \times 5 = 20$
- $2 \times 11 = 22$
- $2 \times 2 \times 11 = 44$
- $5 \times 11 = 55$
- $2 \times 5 \times 11 = 110$
- $2 \times 2 \times 5 \times 11 = 220$

Now show that 220 is not a perfect number.

6. Is 284 a perfect number?
Do you notice anything interesting about 220 and 284 ?
7. Is 496 a perfect number?

Extension

Write a computer program to find perfect numbers.

ACTIVITY 2.3.1

Public Key Ciphers 1

There has always been an interest in codes (or ciphers) and code breaking, but the availability of the internet for quick and efficient communication means that effective coding of messages is now more vital than ever.

Mathematicians have devised a system where it is necessary to keep secret only the decoding key, and not the key used for coding. Using this system, anyone can code a message but only the receiver (or anyone who is informed by the receiver) can decode the message.

An illustration of the method is given on Activity Sheet 2.3.2

1. Following the method on Activity Sheet 2.3.2, code and then decode the word

NORTH

2. (a) Code and decode *NORTH* again as in question 1, but this time use a different pair of *small* prime numbers.
(b) The code given on Activity Sheet 2.3.2 can only be used to code an alphabet of at most 10 letters.

Can you see why?

Extension

Design a similar cipher, using two prime numbers sufficiently large that *all* letters can be coded (and decoded).

What difficulties do you meet using your code?

ACTIVITY 2.3.2

Public Key Ciphers 2

1. Choose two prime numbers, p and q	$p = 2, q = 5$
2. Let $m = p \times q$	$m = 2 \times 5 = 10$
3. Let $A = (p - 1) \times (q - 1)$	$A = 1 \times 4 = 4$
4. Choose a number, C , which is not equal to 1 and is less than A and has no factors in common with A	$C = 3$ (this is the only possible number)
5. Find a number, D , so that $(C \times D) - 1$ is a multiple of A	Since $(3 \times 7) - 1 = 20$ is a multiple of A , take $D = 7$
6. C and m are published; D is kept secret	Publish $C = 3$ and $m = 10$, but keep $D = 7$ secret

To show how it works, consider an 'alphabet' of 9 letters, represented by the digits 1 to 9.

A	D	E	H	N	O	R	S	T
1	2	3	4	5	6	7	8	9

The procedure is shown below to code the word DOOR:

1. Write out message	D	O	O	R
2. Change to numbers according to the table above	2	6	6	7
3. Raise each number to the power $C(=3)$	$2^3 = 8$	$6^3 = 216$	216	$7^3 = 343$
4. Find the remainder on division by $m(=10)$	8	6	6	3

The coded message is now given by 8663.

For decoding, follow this procedure – but you must have knowledge of the secret value of D .

1. Write out message	8	6	6	3
2. Raise to power of $D(=7)$	$8^7 = 2\,097\,152$	279 936	279 936	2187
3. Find remainder on division by $m(=10)$	2	6	6	7
4. Change back to letters according to the table	D	O	O	R

ACTIVITIES 2.1 and 2.2

Notes for Solutions

Notes and solutions given only where appropriate.

2.1 Completed Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Extension

2. You need a 20×20 grid, and multiples of any prime number less than 20 need to be shaded.

2.2 1. 1, 2, 3, 6

$1 + 2 + 3 = 6$, so 6 is a perfect number.

2.

<i>Number</i>	<i>Factors</i>	<i>Sum of Factors (excluding number)</i>
2	1, 2	1
3	1, 3	1
4	1, 2, 4	3
5	1, 5	1

The numbers 2 to 5 are not perfect.

3. For a prime number, the sum of the factors excluding the number will always be 1, so no primes can be perfect.

ACTIVITY 2.2

Notes for Solutions

2.2 4.

<i>Number</i>	<i>Factors</i>	<i>Sum of Factors (excluding number)</i>
7	1, 7	1
8	1, 2, 4, 8	7
9	1, 3, 9	4
10	1, 2, 5, 10	8
11	1, 11	1
12	1, 2, 3, 4, 6, 12	16
13	1, 13	1
14	1, 2, 7, 14	10
15	1, 3, 5, 15	9
16	1, 2, 4, 8, 16	15
17	1, 17	1
18	1, 2, 3, 6, 9, 18	21
19	1, 19	1
20	1, 2, 4, 5, 10, 20	22
21	1, 3, 7, 21	11
22	1, 2, 11, 22	14
23	1, 23	1
24	1, 2, 3, 4, 6, 8, 12, 24	36
25	1, 5, 25	6
26	1, 2, 13, 26	16
27	1, 3, 9, 27	13
28	1, 2, 4, 7, 14, 28	28
29	1, 29	1
30	1, 2, 3, 5, 15, 30	26

So 28 is the next perfect number after 6.

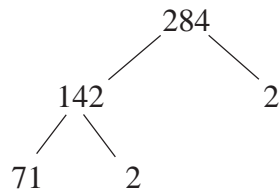
5. $1 + 2 + 5 + 11 + 4 + 10 + 20 + 22 + 44 + 55 + 110 = 284$

So 220 is not a perfect number.

ACTIVITIES 2.2 and 2.3

Notes for Solutions

6.



$$1 + 2 + 71 + 4 + 142 = 220$$

So 284 is not a perfect number.

Note that the sum of the factors of 284 = 220
and the sum of the factors of 220 = 284

The numbers 220 and 284 are called *amicable* numbers, and regarded as tokens of love (use them in your next Valentine's Day card!)

Factors of 284

1

2

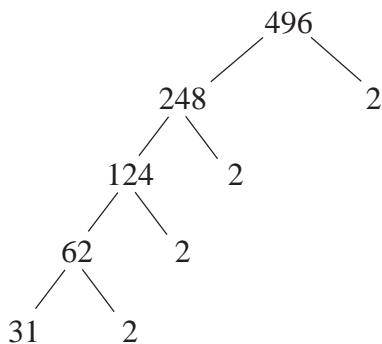
71

$$2 \times 2 = 4$$

$$2 \times 71 = 142$$

$$2 \times 2 \times 71 = 284$$

7.



Factors of 496

1

2

31

$$2 \times 2 = 4$$

$$2 \times 2 \times 2 = 8$$

$$2 \times 2 \times 2 \times 2 = 16$$

$$2 \times 31 = 62$$

$$2 \times 2 \times 31 = 124$$

$$2 \times 2 \times 2 \times 31 = 248$$

$$2 \times 2 \times 2 \times 2 \times 31 = 496$$

$$1 + 2 + 31 + 4 + 8 + 16 + 62 + 124 + 248 = 496$$

So 496 is a perfect number.

2.3 1. 5 6 3 9 4

2. (b) The remainder on division by 10 can take only the values 0, 1, 2, ..., 9; i.e., 10 distinct values. Hence only 10 letters at most can be coded and decoded.